Playing competitive games better with (separable) quantum states

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Everybody knows that entanglement is a resource. E.g., it can help distant, non-interacting players to outperform only classically correlated ones:

$$\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle \leq 2$$

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$2 \to 2\sqrt{2}$

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Here I want to show you that separable correlations can provide a game advantage, as long as they are nonclassical (even classical-quantum states with non-zero discord)

Wait – discord!?

0. Games

Board, card, strategy, sport, ...: games don't have to be solely for entertainment, there are also military and economic games.

There don't necessarily have to be rules, but each player needs to have a set of moves and needs to know what their winning objective is.

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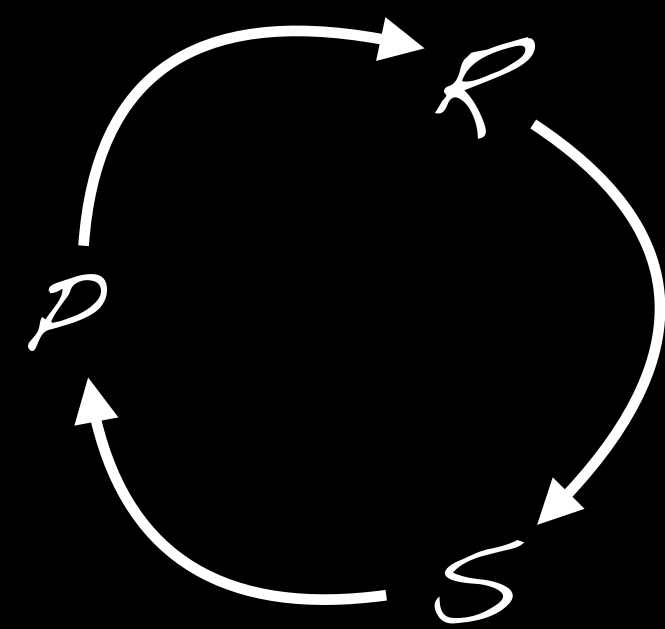
Chess, checkers, tic-tac-toe, ...: each player has an optimal pure strategy – equilibrium

0. Games

Example: Rock-Paper-Scissors



Does not have a dominant strategy.

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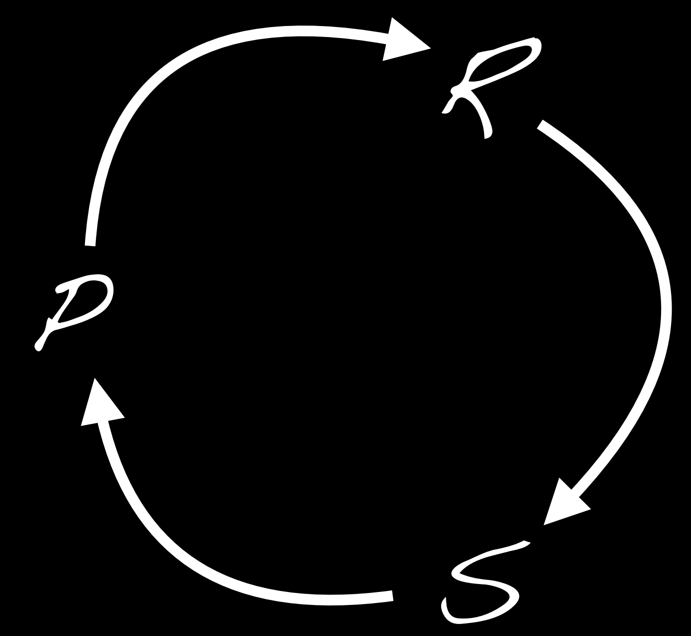

A \ B	R	P	S
R	0,0	-1,+1	+1,-1
P	+1,-1	0,0	-1,+1
S	-1,+1	+1,-1	0,0

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Optimal play is to behave randomly: expected payoff 0 to both players.

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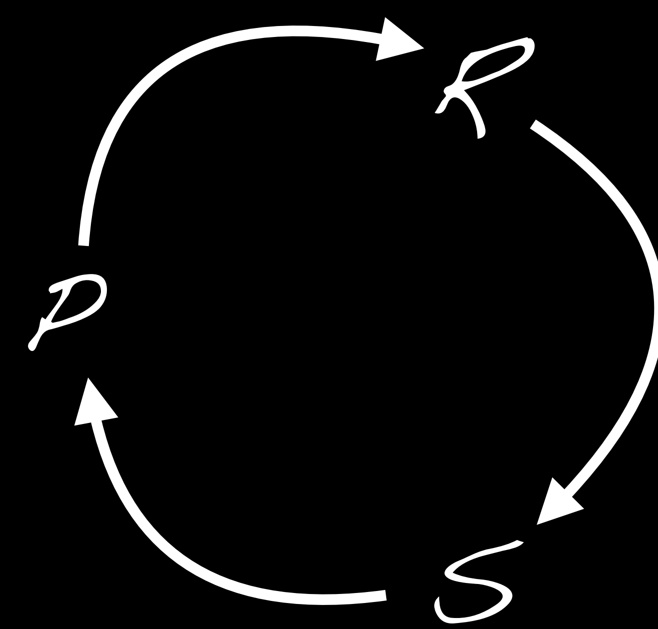

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Optimal play is to behave randomly: expected payoff 0 to both players.

Zero-sum game!

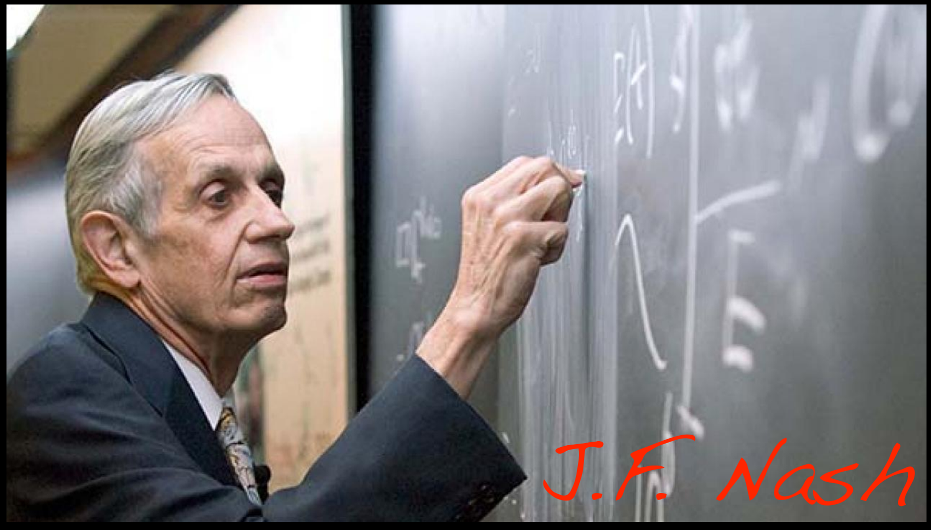
Minimax thm.: $\exists$ equilibrium in mixed strategies

0. Games

Example: Rock-Paper-Scissors with penalty

A \ B	R	P	S
R	−9, −9	−1, +1	+1, −1
P	+1, −1	−9, −9	−1, +1
S	−1, +1	+1, −1	−9, −9

No longer zero-sum, but Nash's theorem guarantees equilibrium in mixed strategies. Uniformly random still optimal



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With correlation advice $(A,B) \in \{RP, RS, PR, PS, SR, SP\}$ players avoid the penalty. Correlated equilibrium: no incentive to deviate unilaterally

0. Games

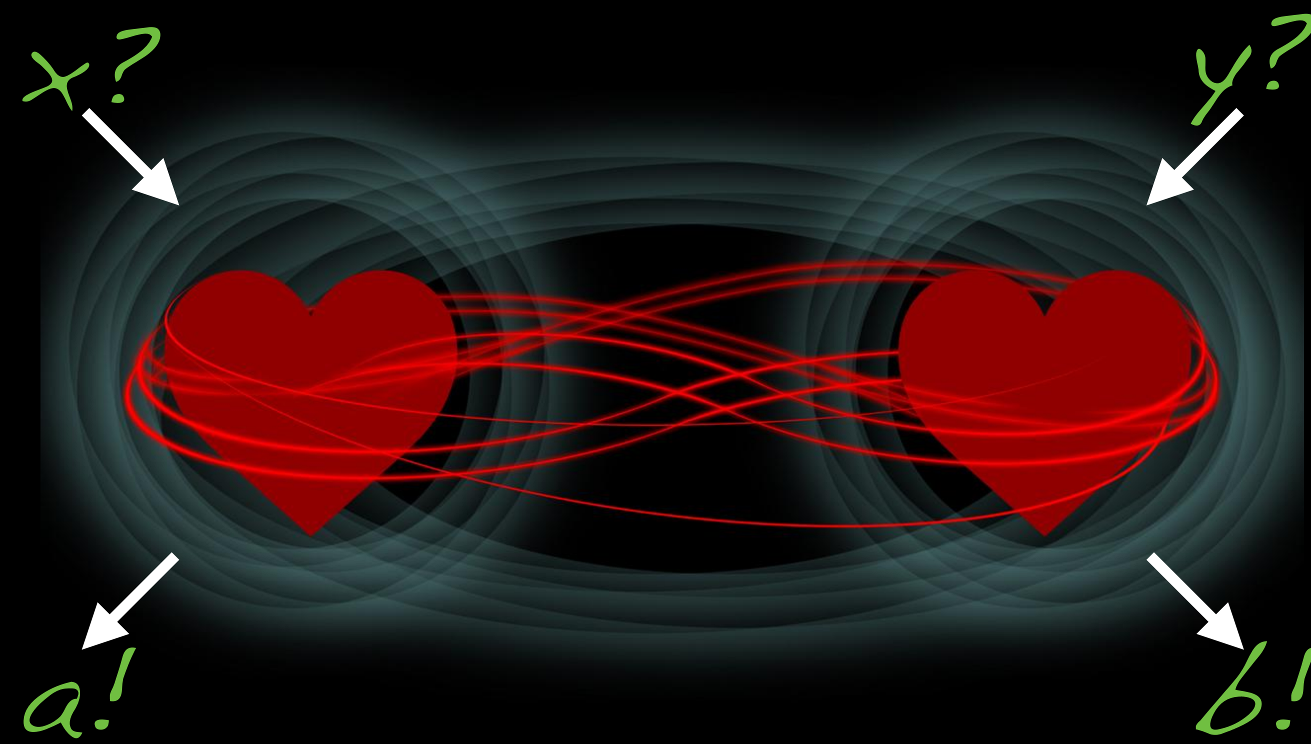
Could quantum correlations provide any even better equilibria?

Not in games of complete information as the above: since each player has to pick a strategy, the quantum state must be measured. And the resulting joint distribution is a correlated equilibrium à la Aumann.

[S. Zhang, Proc. ICTS 2012]

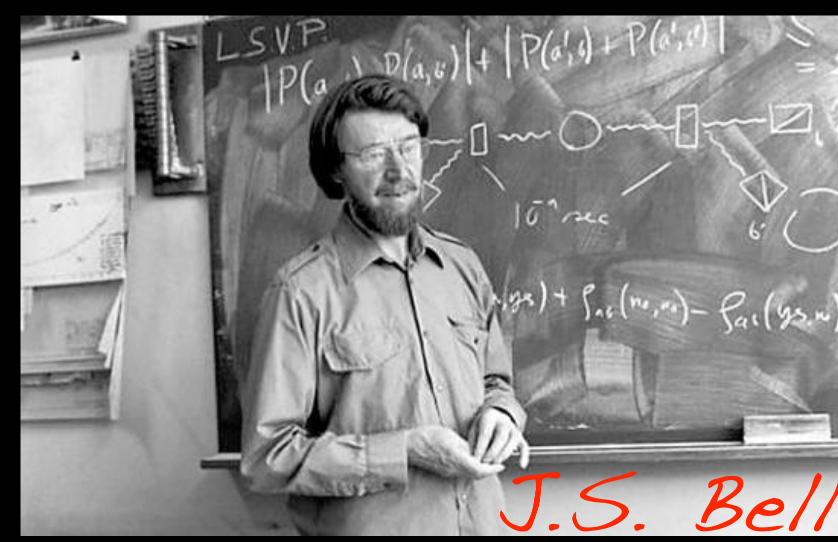
1. Nonlocal games

...are cooperative games: two or more players try to pass interrogation by a referee (who doesn't have an objective).



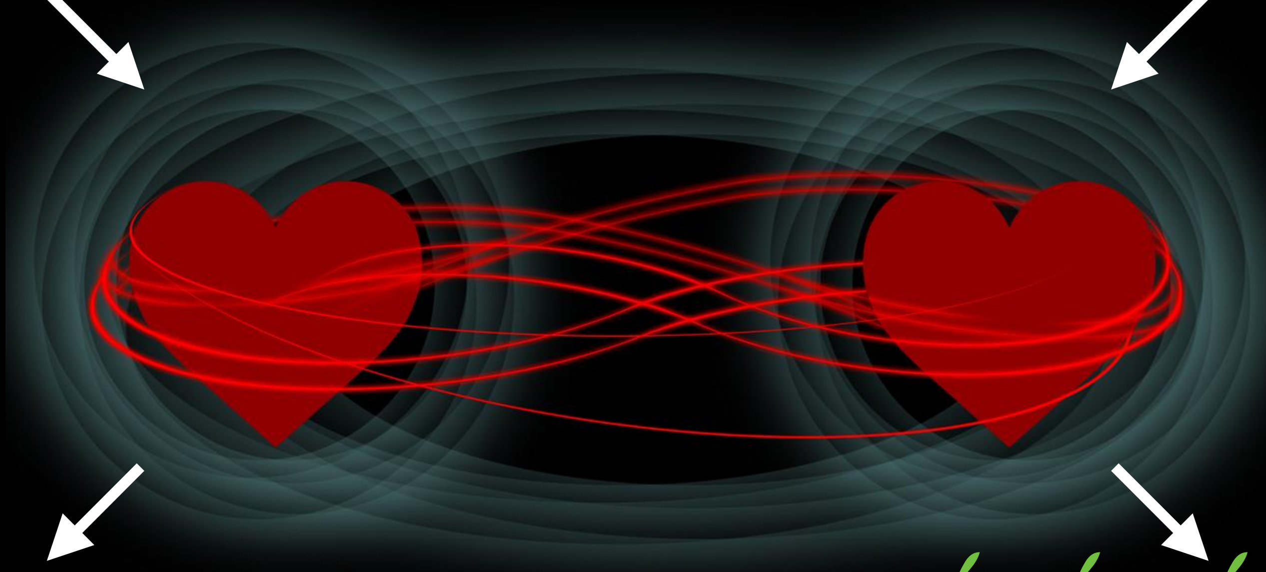
Correlation clearly helps ("crib sheet").

Bell: Entanglement can help more, even allowing to win with certainty a game that can't be won classically ("pseudo-telepathy")



Magic square game [Peres & Mermin]

$r \in \{1,2,3\}$ ↘ ↙ $c \in \{1,2,3\}$



↙ $a_1, a_2, a_3 \in \{\pm 1\}$ ↘ $b_1, b_2, b_3 \in \{\pm 1\}$

$r=2,\ c=1$

S_{11}	S_{12}	S_{13}	→ -1
S_{21}	S_{22}	S_{23}	→ -1
S_{31}	S_{32}	S_{33}	→ -1
↓ $+1$	↓ $+1$	↓ $+1$	

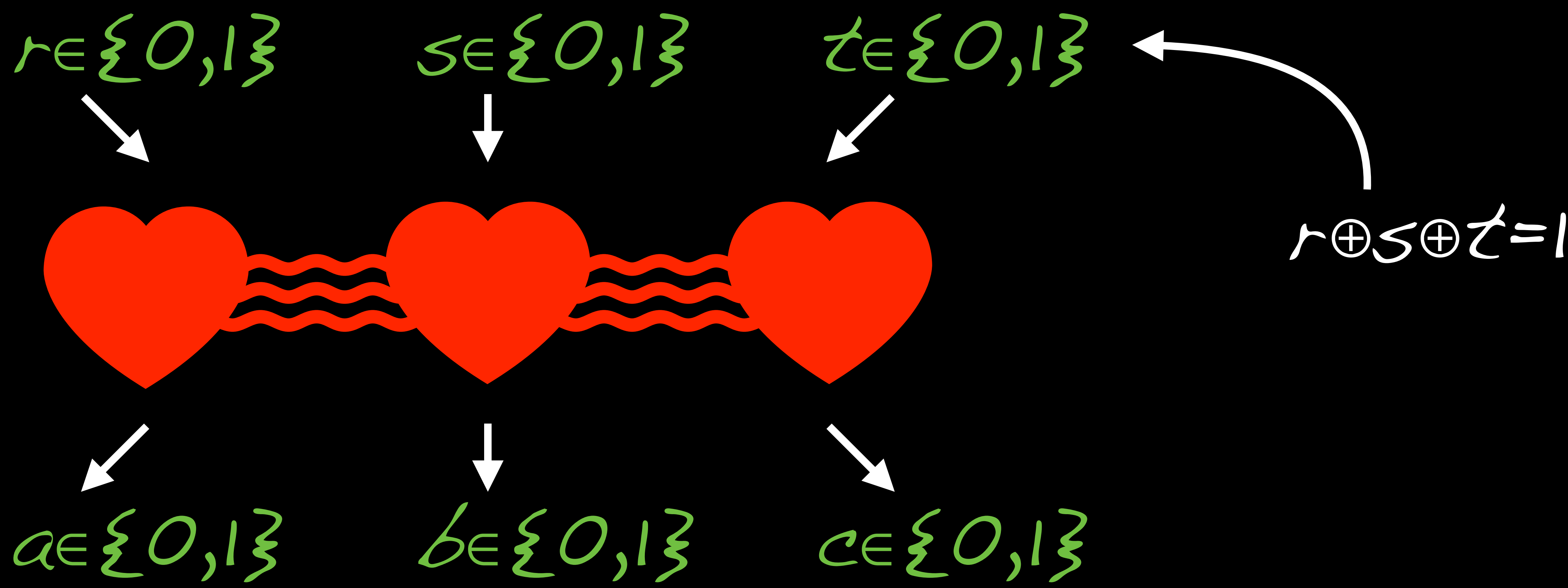
Win: $a_1 a_2 a_3 = -1$,

$b_1 b_2 b_3 = +1$,

& $a_c = b_r$

$p_C = 8/9,\ p_Q = 1$

GHZ game [Greenberger/Horne/Zeilinger]



Win iff $a \oplus b \oplus c \oplus 1 = rst$

$P_C = 3/4$, $P_Q = 1$

2. Bayesian games and equilibria

A Bayesian game [Harsányi 1968] is just like a nonlocal game, with inputs (types) t_i and outputs (actions) a_i for each player ($i=1,\ldots,n$); $T=T_1\ldots T_n \sim \pi$ a given p.d. But now each player i has their own payoff function $u_i(t,a) \in \mathbb{R}$.

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Also called games of incomplete info, because t determines the actual game, but each player knows only partially which one is played.

[von Neumann/Nash: π = point mass]

2. Bayesian games and equilibria

Can still apply von Neumann-Nash framework: strategies now are functions, $\mathbb{F}_i := \{f_i : t_i \to a_i\}$.

Def. A correlated equilibrium is a collection of random functions $F_i \in \mathbb{F}_i$, such that for every j and all Markov chains $F_j' - F_j - F_{[n]\setminus j}$,

$$\mathbb{E}u_j(T, A) \geq \mathbb{E}u_j(T, A_{[n]\setminus j}, A_j'),$$

where $A_i = F_i(T_i)$ for all $i \in [n]$, $A_j' = F_j'(T_j)$.

[Cf. Maschler/Solan/Zamir, Game Theory, 2013]

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Nash equilibrium if the F_i are all independent.

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Strategy defines a behaviour

$$Q(a_1 \ldots a_n | t_1 \ldots t_n) = \Pr\{\forall i \;\; a_i = F_i(t_i)\},$$

which is local for correlated equilibria and a product channel $Q_1(a_1|t_1) \ldots Q_n(a_n|t_n)$ for Nash.

[Cf. Maschler/Solan/Zamir, Game Theory, 2013]

For a given game G, consider

$\mathrm{CorrE}(G) := \{Q$ behaviour of a corr. equi.$\}$,

so that

$\mathrm{NE}(G) := \{Q$ behaviour of a Nash equi.$\}$

$= \mathrm{CorrE}(G) \cap \{$product channels$\} \neq \emptyset$.

Fact: $\mathrm{CorrE}(G)$ is a convex polytope. That's because equilibrium conditions captured by finitely many linear inequalities on the joint distribution of $F_1 \ldots F_n$.

[Cf. Maschler/Solan/Zamir, Game Theory, 2013]

Def. A *quantum correlated equilibrium* consists of an *n-party state* ρ and *POVMs* $(M^{t_i}_{a_i})$, such that for every player j and *any other POVM* $(M'^{t_j}_{a_j})$,

$$\mathbb{E}u_j(T,A) \geq \mathbb{E}u_j(T, A_{[n]\setminus j}A_j'),$$

with respect to the quantum behaviours

$$Q(a_1 \ldots a_n | t_1 \ldots t_n) = \mathrm{Tr}\, \rho(M^{t_1}_{a_1} \otimes \ldots \otimes M^{t_n}_{a_n}),$$

$$Q'(a_{[n]\setminus j} a_j' | t_1 \ldots t_n) = \mathrm{Tr}\, \rho(M^{t_1}_{a_1} \otimes \ldots \otimes M'^{t_j}_{a_j} \otimes \ldots \otimes M^{t_n}_{a_n}).$$

[Auletta et al., 1605.07896]

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$QuE(G) := \{Q$ behaviour of a qu. corr. equi.$\}$ is convex alright, but generally not a closed polytope...

[Auletta et al., 1605.07896]

Note: Behaviour Q plus $T \sim \pi$ define a joint distribution $\Pr\{T=t, A=a\} = \pi(t)Q(a|t)$, hence expected payoffs $\mathbb{E}u_i(T,A)$ are linear functions of Q. (But the various equilibrium conditions are not property of Q!)

Thus, to witness separations between

$NE(G) \subseteq conv(NE(G)) \subseteq CorrE(G) \subseteq QuE(G)$,

we use linear functions, such as the social welfare $s = \frac{1}{n} \sum_{i=1}^{n} u_i$ and its expectation.

3. Quantum advantage: entanglement and separable states

Example: Modified GHZ [for $0<u(0)<u(1)$],
uniform $(r,s,t) \in \{001,010,100,111\}$,
winning predicate $V(r,s,t,a,b,c) = \delta(a\oplus b\oplus c\oplus 1, rst)$

Let
$$u_A(r,s,t,a,b,c) := u(a)V(r,s,t,a,b,c),$$
$$u_B(r,s,t,a,b,c) := u(b)V(r,s,t,a,b,c),$$
$$u_C(r,s,t,a,b,c) := u(c)V(r,s,t,a,b,c),$$
i.e. players win and lose together, but amount depends on their actual output

[M. Cerdà Ramon, BSc thesis UAB, 2021]

Example: Fix $0 < u(0) < u(1)$, GHZ winning predicate $V(r,s,t,a,b,c) = \delta(a \oplus b \oplus c \oplus 1, rst)$, let $u_A(r,s,t,a,b,c) := u(a)V(r,s,t,a,b,c)$, etc.

Thm. Advice state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and GHZ measurements X,Y is quantum corr. equilibrium with $\mathbb{E}u_{A/B/C} = \mathbb{E}s = \frac{1}{2}(u(0)+u(1))$.

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For any class. correlated equilibrium, largest social welfare $\mathbb{E}s = \frac{3}{4}u(1)$, so have quantum advantage for $u(0)>\frac{1}{2}u(1)$.

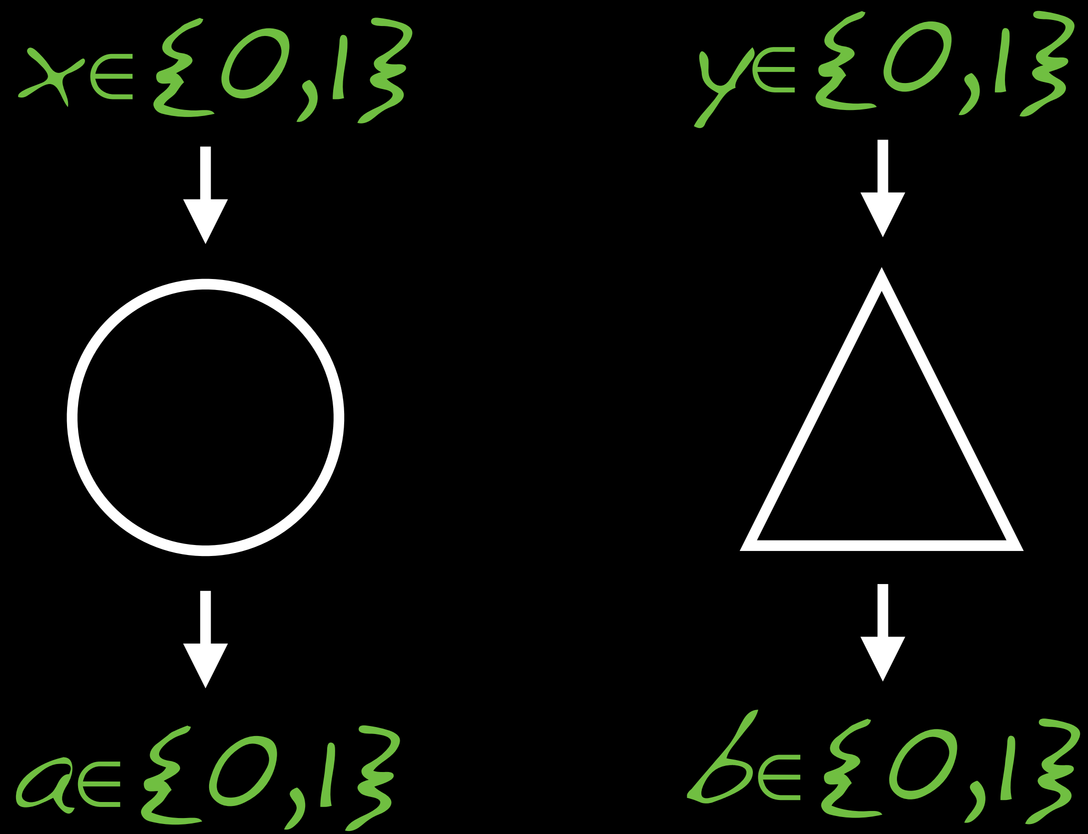
[M. Cerdà Ramon, BSc thesis UAB, 2021]

Remark: We are cheating considerably, as not only do we use the entangled state and measurements from the quantum strategy, but the bound on the classical social welfare is obtained by maximising over all local strategies (not just equilibria).

All prior examples of quantum advantage are just dressed-up Bell inequalities, too!

[La Mura 2005; Pappa et al. 2015; Auletta et al. 2016; Rai et al. 2016; Bolonek-Lasoń 2017; ...]

Example: CHSH



Winning predicate:

$$V(a,b,x,y) = \delta(a\oplus b, xy)$$

$$\max_{\text{class.}} \Pr\{\text{win}\} = 3/4$$

$$\max_{\text{ent.}} \Pr\{\text{win}\} = \cos^2 \pi/8$$

State $|\Phi\rangle$;

Alice: X, Z; Bob: $\tilde{X}, \tilde{Z}$

[Clauser/Horne/Shimony/Holt, PRL 1968]

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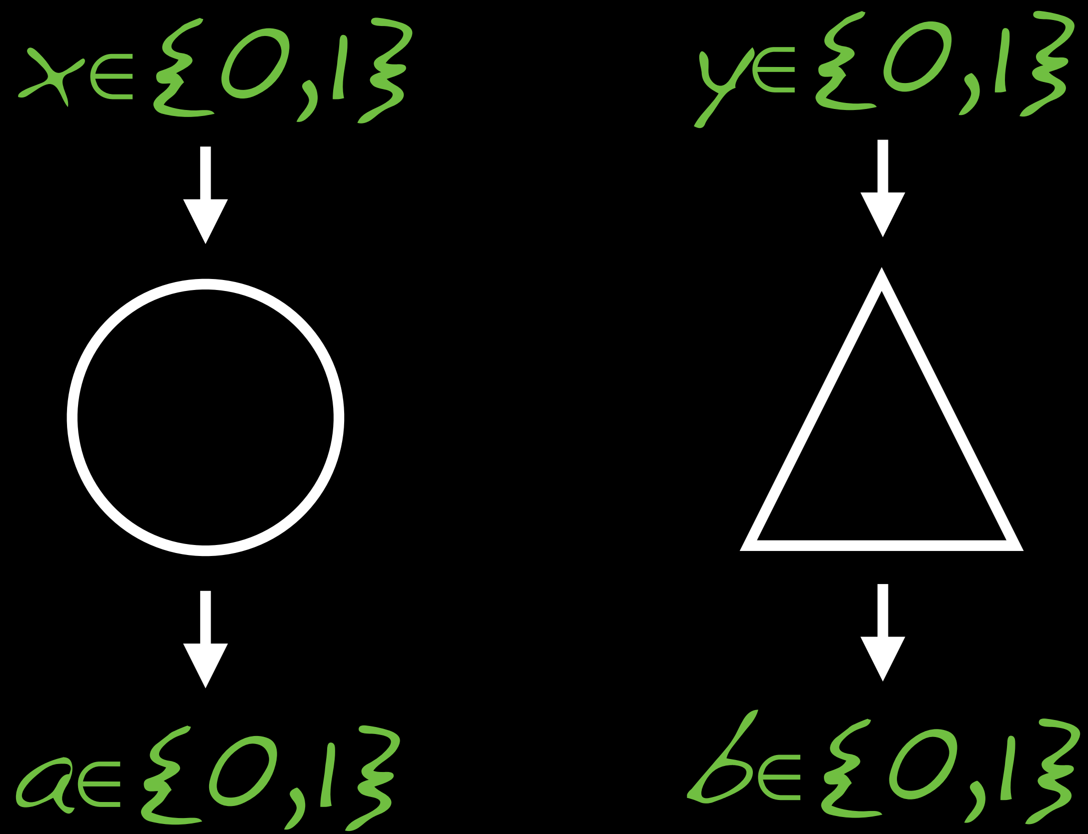

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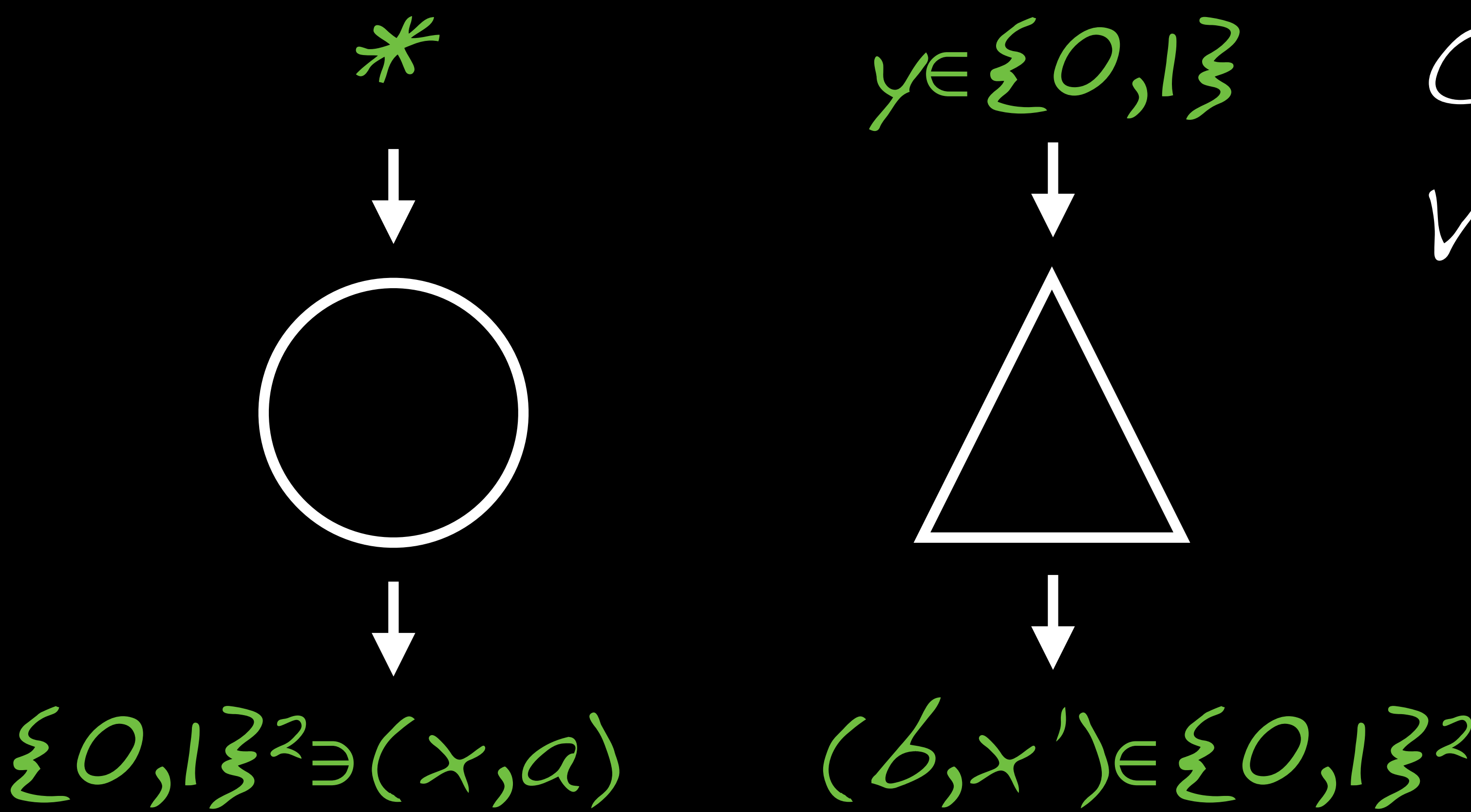
As in GHZ, can modify this by introducing modest competition, so that $3/4$ and $\cos^2 \pi/8$ are largest classical and quantum social welfare, resp...

[Pappa et al., PRL 2015]

Example: CHSH for capitalists



CHSH winning predicate:

$$V(x,y,a,b) = \delta(a \oplus b, xy)$$

$$u_A(y,x,a,b,x') := V(x,y,a,b) - \lambda(2\delta_{xx'} - 1)$$

$$u_B(y,x,a,b,x') := V(x,y,a,b) + \lambda(2\delta_{xx'} - 1)$$

[Wang/Scarpa/AW, in preparation, 2026]

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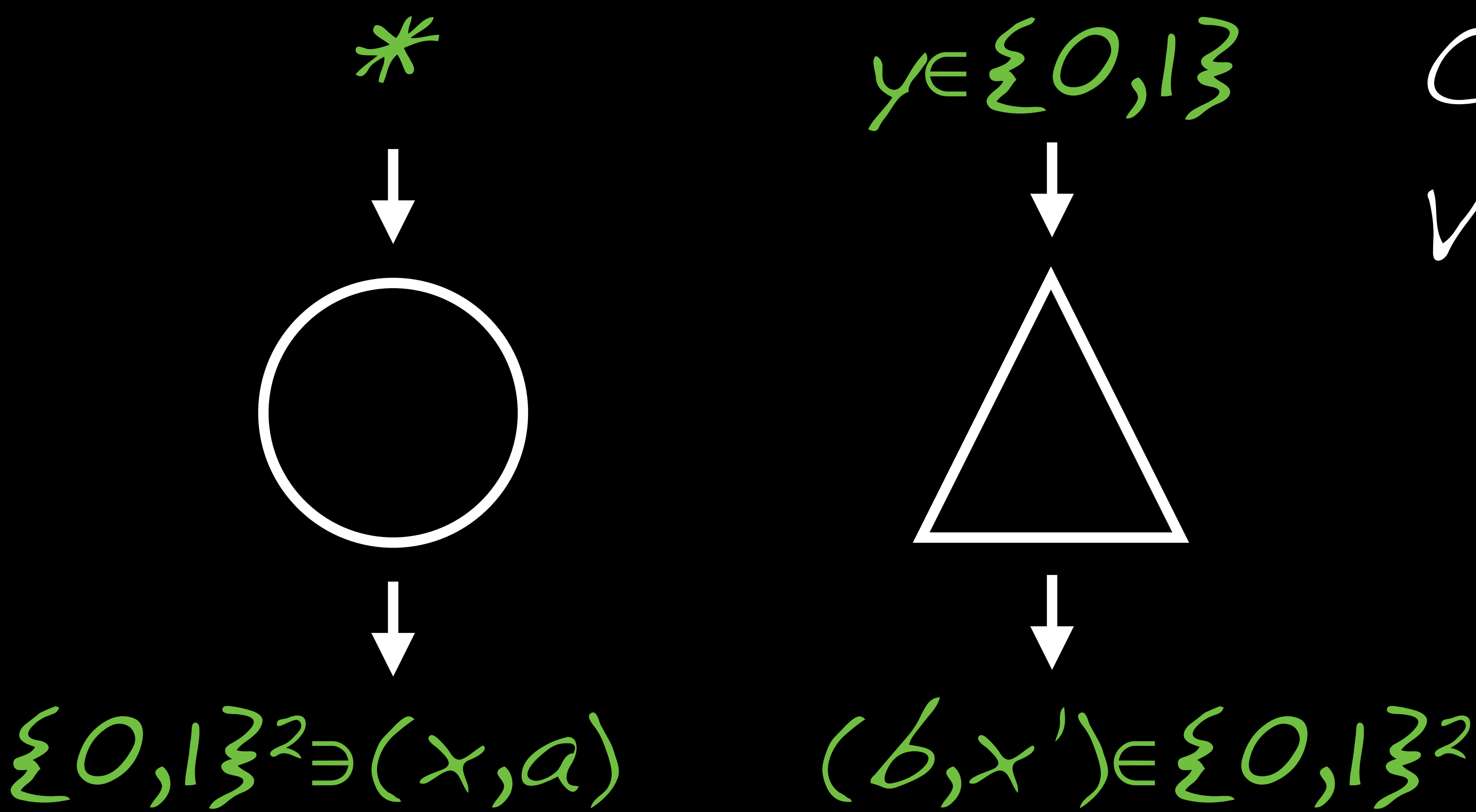

CHSH winning predicate:

$V(x,y,a,b) = \delta(a \oplus b, xy)$

Incentivises Alice to output uniform x

$$u_A(y,x,a,b,x') := V(x,y,a,b) - \lambda(2\delta_{xx'} - 1)$$

$$u_B(y,x,a,b,x') := V(x,y,a,b) + \lambda(2\delta_{xx'} - 1)$$

I.e., CHSH with Alice choosing x; Bob is rewarded (Alice punished) if he guesses x'=x

[Wang/Scarpa/AW, in preparation, 2026]

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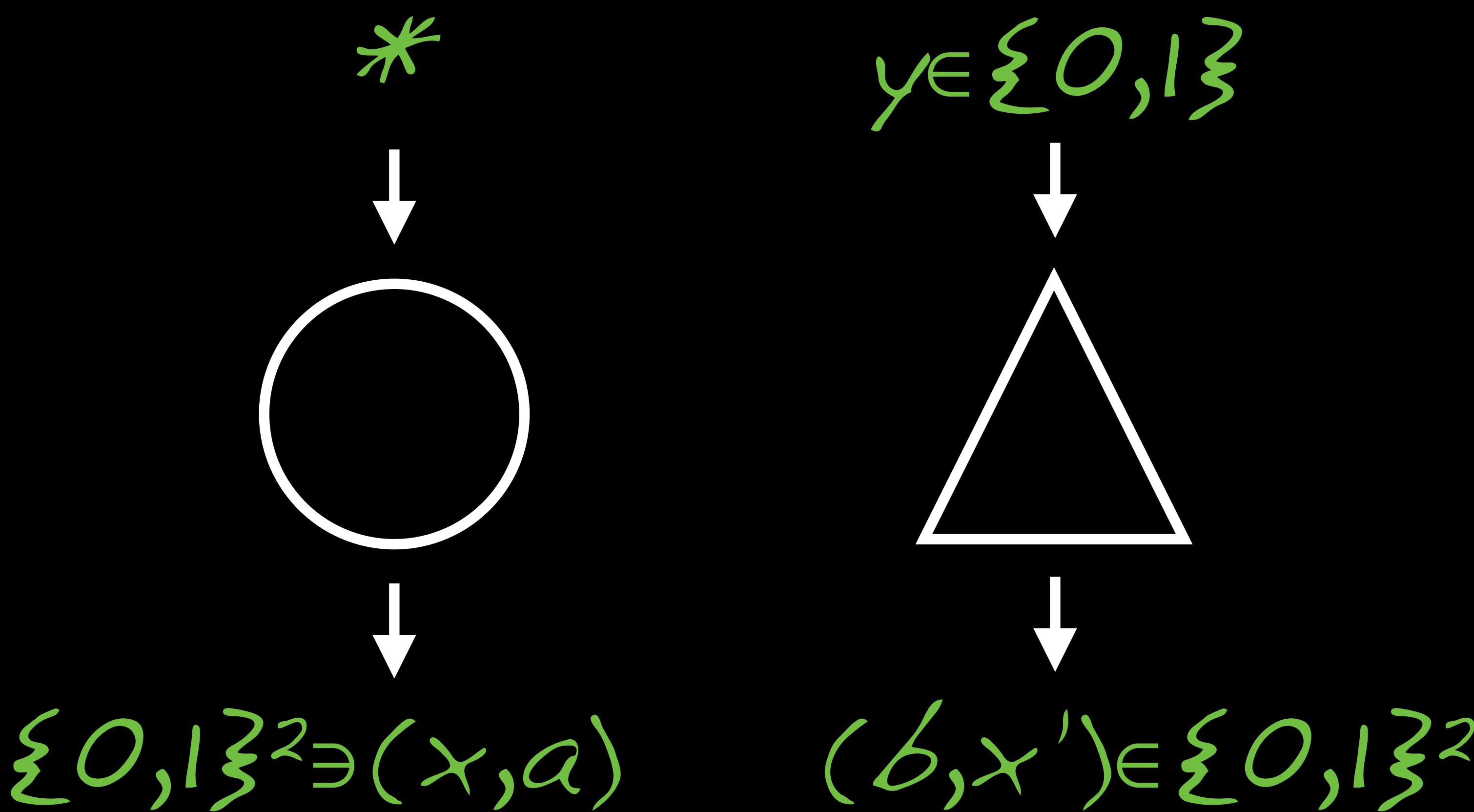

$$u_{A/B}(y,x,a,b,x') := V(x,y,a,b) \mp \lambda(2\delta_{xx'}-1)$$

Thm. 1: Generating x & x' uniformly and using optimal classical strategy, is a correlated equilibrium with $\mathbb{E}u_{A/B} = \mathbb{E}s = 3/4$.

[Wang/Scarpa/AW, in preparation, 2026]

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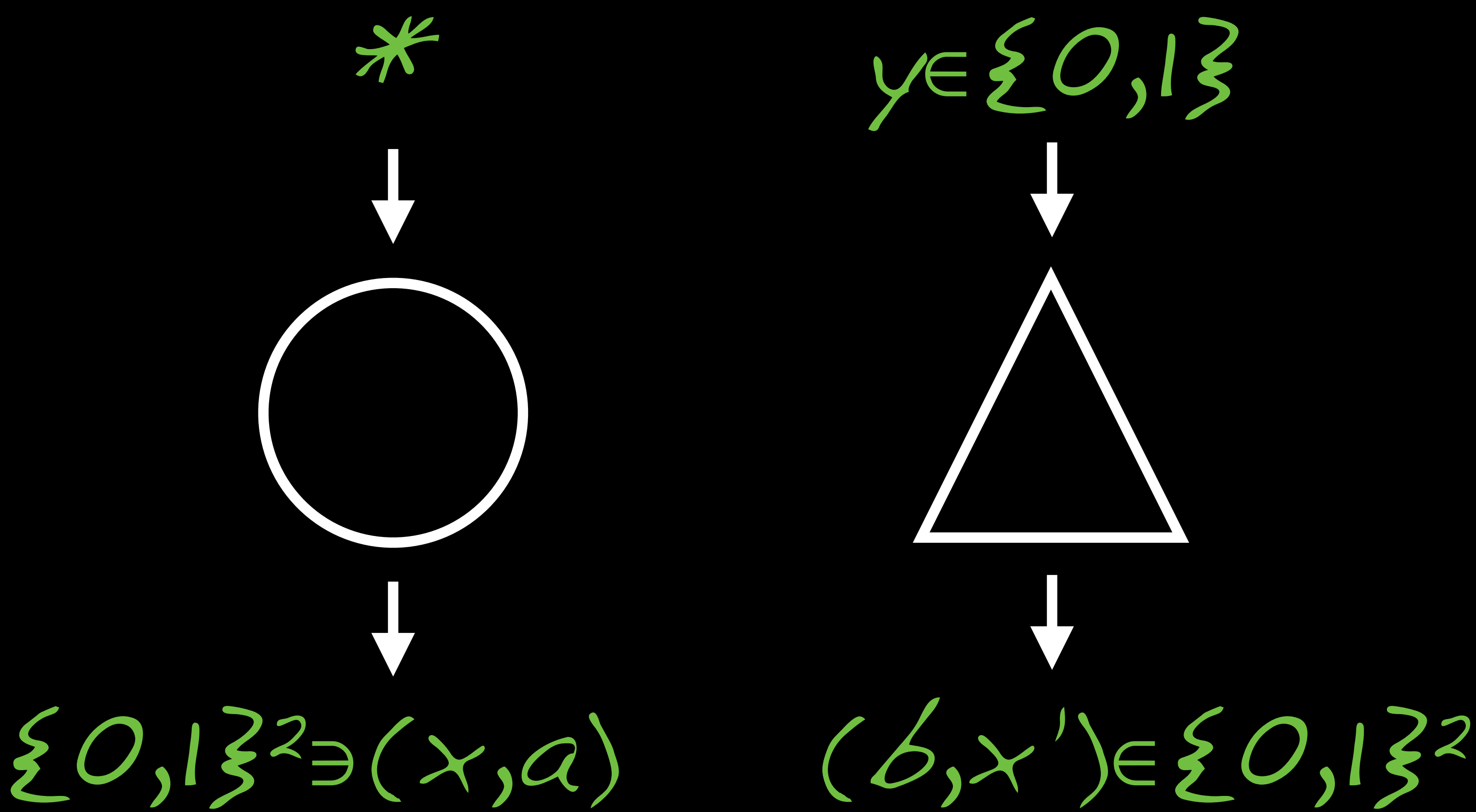

$$u_{A/B}(y,x,a,b,x') := V(x,y,a,b) \mp \lambda(2\delta_{xx'}-1)$$

Thm. 1 cont'd: For λ large enough, every classically correlated equilibrium has social welfare $\mathbb{E}s \leq 3/4$.

[Wang/Scarpa/AW, in preparation, 2026]

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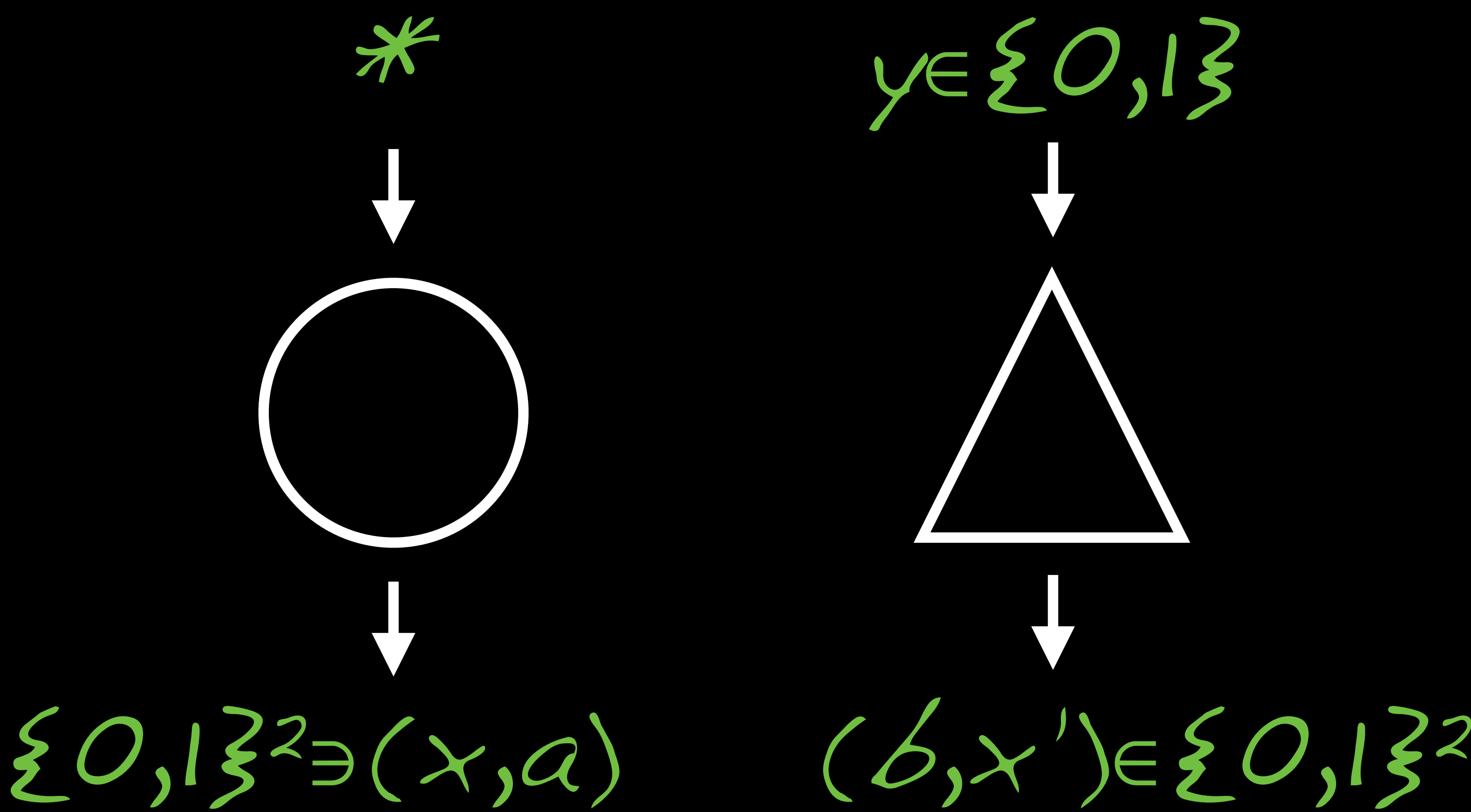

$$u_{A/B}(y,x,a,b,x') := V(x,y,a,b) \mp \lambda(2\delta_{xx'}-1)$$

Thm. 2: Generating x & x' uniformly and using optimal CHSH strategy, is a quantum corr. equilibrium with $\mathbb{E}u_{A/B} = \mathbb{E}s = \cos^2 \pi/8$.

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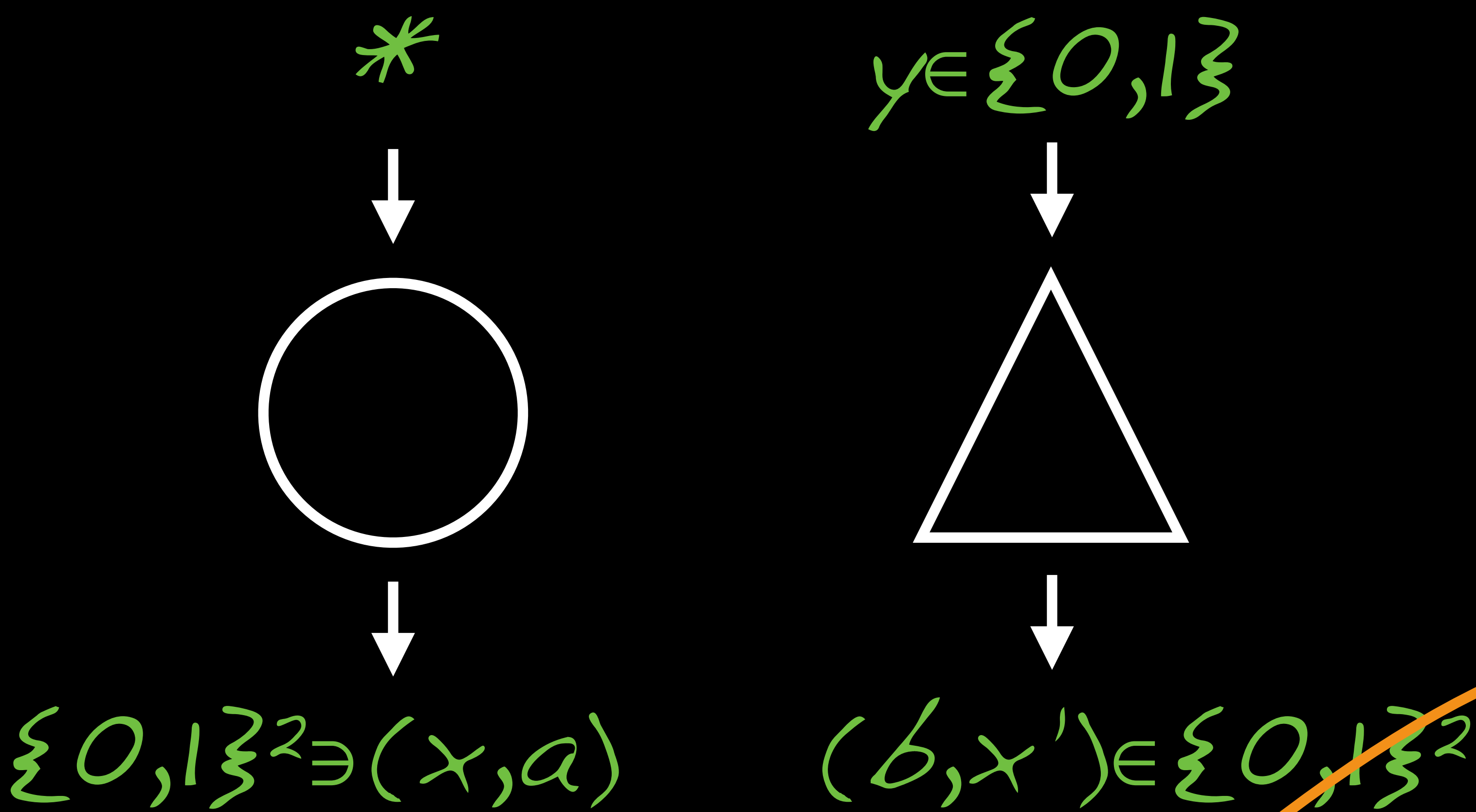

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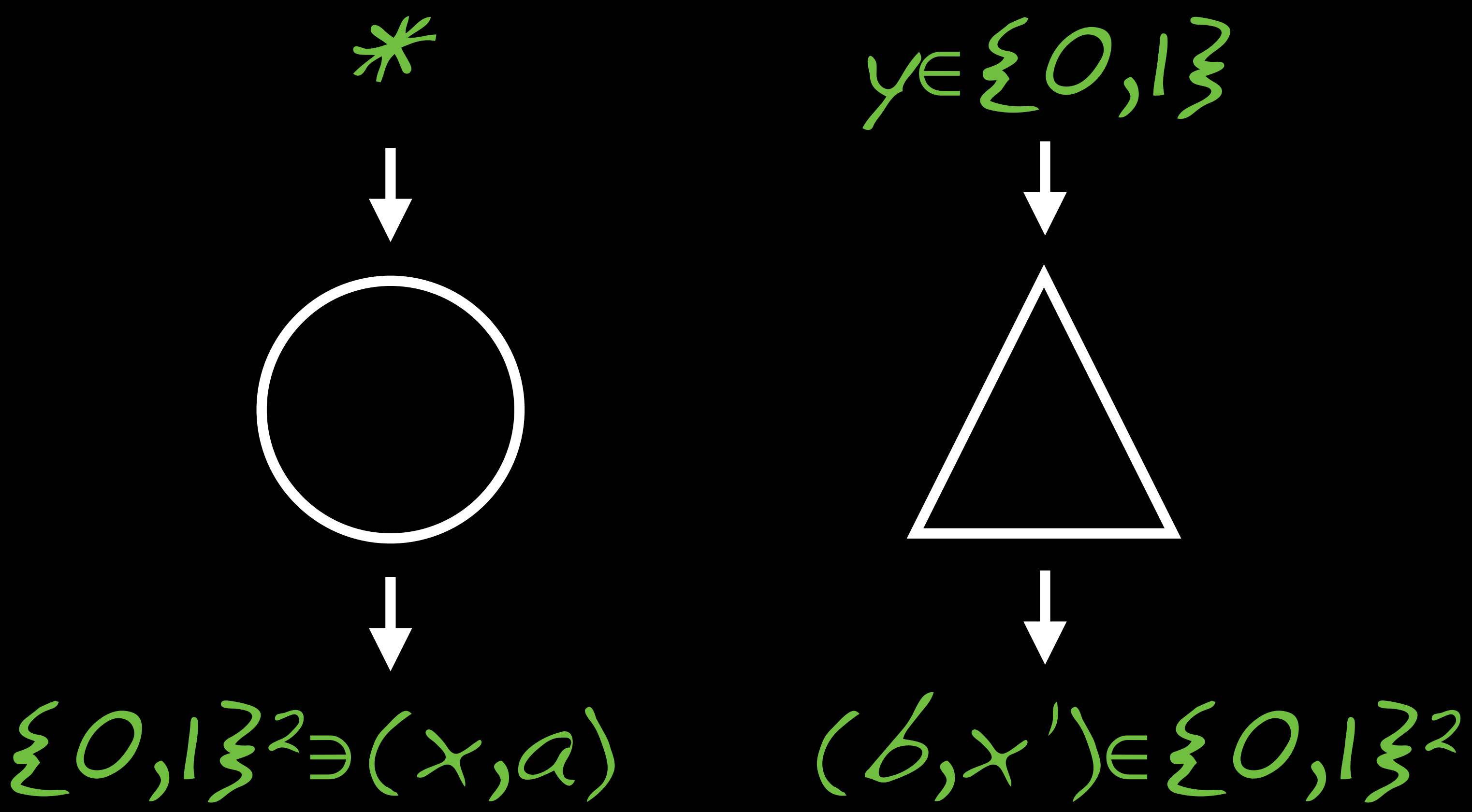

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$:= V(x,y,a,b) \mp \lambda(2\delta_{xx'}-1)$

Since x is uniformly generated, might as well already carry out Alice's CHSH measurement:

$$\omega = \frac{1}{8} \sum |xa\rangle\langle xa|^A \otimes \left(|\psi_{xa}\rangle\langle\psi_{xa}| \otimes |x'\rangle\langle x'|\right)^B$$

[Wang/Scarpa/AW, in preparation, 2026]

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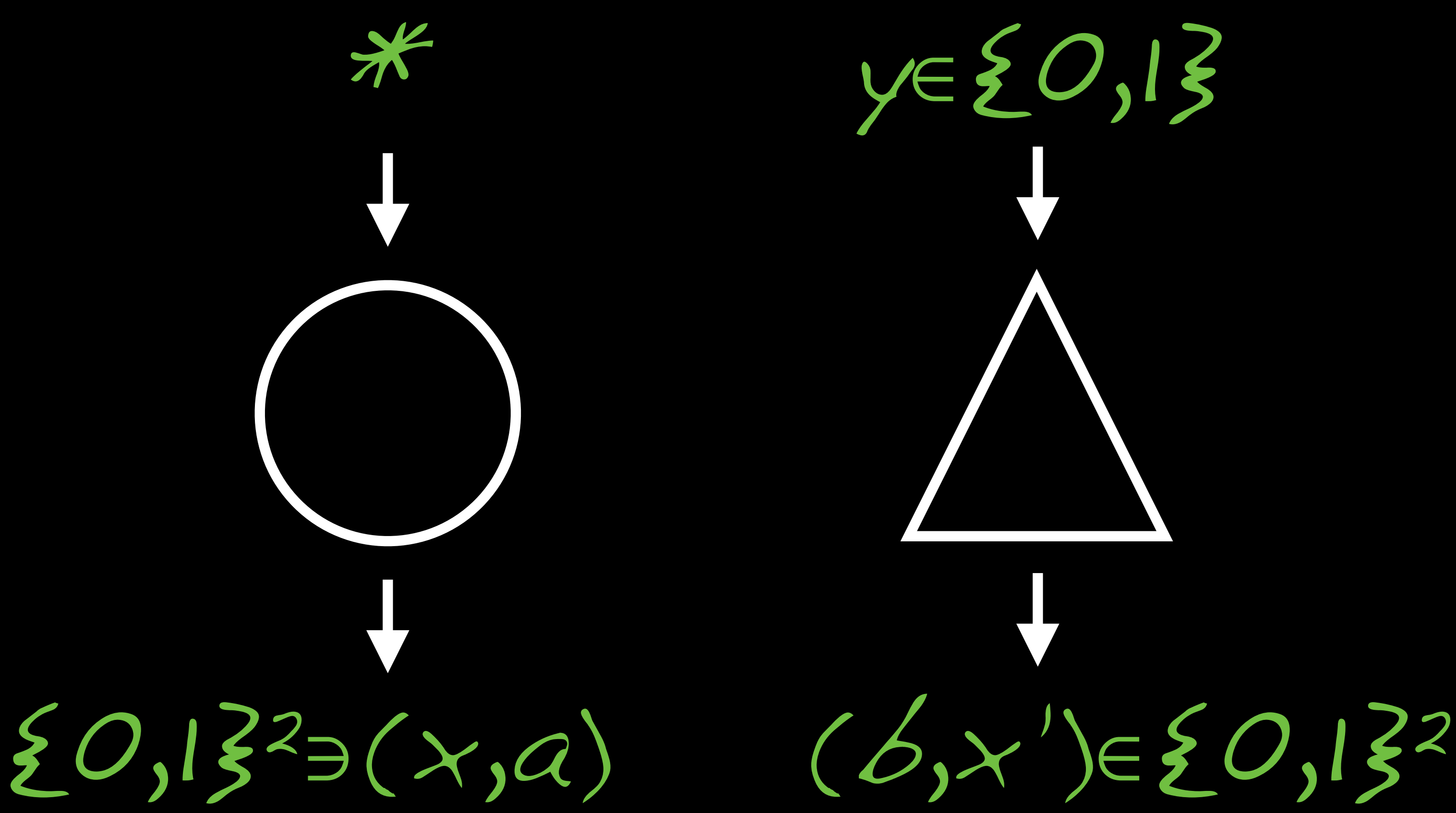

$$u_{A/B}(y,x,a,b,x') := V(x,y,a,b) \mp \lambda(2\delta_{xx'}-1)$$

Thm. 2+: Advice state ω and Bob's CHSH measurements $\tilde{X},\tilde{Z}$, is a quantum correlated equilibrium with $\mathbb{E}u_{A/B} = \mathbb{E}s = \cos^2 \pi/8$.

[Wang/Scarpa/AW, in preparation, 2026]

4. Conclusion & open questions

Same method can be applied to any nonlocal game G to get G-for-capitalists: optimal entangled strategy gives rise to quantum corr. equilibrium with a separable, indeed classical-quantum state. If G is a pseudo-telepathy game (magic square, GHZ, ...), social welfare shows quantum advantage.

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Zero-sum term corresponding to guessing x is an instance of a "mechanism." Here: law enforcement by fines and bounties.

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Behaviour Q of above quantum correlated equilibrium of CHSH-for-capitalists is local. I.e. it's a mixture of random local functions, but none of them is a classically correlated equilibrium.

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Are there real-world games with an entangled or separable quantum advantage?

Shape of quantum equilibrium set? Convex, but in general computationally hard. Are there SDP outer approximations?


Tipi Napoletani - Il giuoco della morra.