Universal Entanglement Distillation

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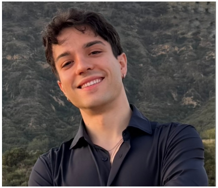
UvA

Coauthors



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Quantum tomography

A *fancy* way to say statistical estimation.

Given many copies of a state ρ we obtain a *classical* estimate $\bar{\rho}$, such that

$$\frac{1}{2}\|\rho - \bar{\rho}\|_1 \leq \varepsilon = O\left(\frac{rd}{\sqrt{n_{\mathrm{copies}}}}\right) . \quad (1)$$

$$\|X\|_1 := \mathrm{Tr}\left[\sqrt{X^\dagger X}\right] . \quad (2)$$

Fidelity

The fidelity is:

$$F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 \ . \tag{3}$$

Alternatively the *infidelity* is:

$$1 - F(\rho,\sigma) \ . \tag{4}$$

The fidelity is linked to the trace-distince via the Fuchs–van de Graaf inequalities

$$1 - F(\rho,\sigma) \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F^2(\rho,\sigma)} \ . \tag{5}$$

Fidelity is *tensor-multiplicative*

LOCC

Local Operations and Classical Communication.

We can act *quantumly* locally on A or B, but we can exchange as much classical information as we want.

E. Chitambar et al., Commun. Math. Phys. **328**, 303-326 (2014)

Entanglement

Quantum supercorrelation.

There is no local way to measure the whole state of the system

Entangled states

$$\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B \ . \tag{6}$$

A state ρ_{AB} is *separable* if

$$\rho_{AB} = \sum_j p_j \rho_j^{(A)} \otimes \rho_j^{(B)} \ . \tag{7}$$

A state σ_{AB} is entangled if it is NOT separable.

Entanglement distillation

Given many copies of ρ_{AB}, we want to employ LOCCs to obtain the largest R, such that

$$\rho_{AB}^{\otimes n} \to \Phi_2^{\otimes nR} \, . \tag{8}$$

LOCC do not create entanglement.

If $R > 0$, then ρ_{AB} is *distillable*.

Entanglement distillation

Given a bipartite state ρ_{AB}, the **distillable entanglement of** ρ_{AB} **at error threshold** $\varepsilon \in [0,1)$ is defined to be

$$E_d^{\varepsilon}(\rho_{AB}) := \sup \Bigg\{ R > 0 : \limsup_{n\to\infty} \inf_{\Lambda_n \in \mathrm{LOCC}\left(A^n B^n \to A_0^{\lceil Rn \rceil} B_0^{\lceil Rn \rceil}\right)} \frac{1}{2} \left\| \Lambda_n\left(\rho_{AB}^{\otimes n}\right) - \Phi_2^{\otimes \lceil Rn \rceil} \right\|_1 \leq \varepsilon \Bigg\} . \quad (9)$$

The **distillable entanglement** of ρ_{AB} and the **strong converse distillable entanglement** of ρ_{AB} are then given by

$$E_d(\rho_{AB}) := E_d^0(\rho_{AB}) , \qquad E_d^{\dagger}(\rho_{AB}) := \sup_{\varepsilon \in [0,1)} E_d^{\varepsilon}(\rho_{AB}) . \quad (10)$$

C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher,. Phys. Rev. A, **53** 2046-2052 (1996).
C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett., **76** 722-725 (1996).

What if the state is classically unknown?

Universal distillation

Given a family of quantum states $\mathcal{X} \in \mathcal{D}(\mathcal{H}_{AB})$ we define a figure of merit that quantifies the maximum amount of distillable entanglement from an unknown state ρ_{AB} belonging to $\mathcal{X}$.

$$\hat{E}_d^{\varepsilon}(\mathcal{X}) := \sup \left\{ R > 0 : \limsup_{n\to\infty} \inf_{\Lambda_n \in \mathrm{LOCC}(A_n B_n \to A_n^0 B_n^0)} \sup_{\rho_{AB}\in\mathcal{X}} \frac{1}{2} \left\| \Lambda_n(\rho_{AB}^{\otimes n}) - \Phi_2^{\otimes \lceil Rn \rceil} \right\|_1 \leq \varepsilon \right\} . \tag{11}$$

We call $\hat{E}_d^{\varepsilon}(\mathcal{X})$ the *universal distillable entanglement* at error ε for the family $\mathcal{X}$. Again we call

$$\hat{E}_d(\mathcal{X}) := \hat{E}_d^{0}(\mathcal{X}) \qquad \hat{E}_d^{\dagger} := \sup_{\varepsilon\in[0,1)} \hat{E}_d^{\varepsilon}(\mathcal{X}) . \tag{12}$$

The protocol

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  $\rho_{AB} \to \mathcal{N}(\rho_{AB})$

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- Estimate ρ_{AB} from $\mathcal{N}(\rho_{AB})$ in Bob's lab
- Distill from the remaining copies given the estimate on the state

Classical teleportation

We perform a quantum teleportation protocol using the separable *isotropic state*

$$W_d := \frac{1}{d+1}\Phi_d + \left(1 - \frac{1}{d+1}\right)\frac{\mathbb{I}}{d^2}\,, \tag{13}$$

where Φ_d is the d-dimensional maximally entangled state. Since W_d is separable, it can be prepared via LOCCs.

Classical teleportation

After the procedure, Bob has the state:

$$\mathcal{N}(\rho_{AB}) = \frac{1}{d+1}\rho_{AB} + \left(1 - \frac{1}{d+1}\right)\frac{\mathbb{I}_A}{d} \otimes \mathrm{Tr}_A[\rho_{AB}]\,. \tag{14}$$

Estimation

B can estimate ρ_{AB} from $\mathcal{N}(\rho_{AB})$.

Let ρ_{AB} be a $d \times d$-dimensional state. Then, there exists a quantum state tomography algorithm that takes as input n copies of ρ_{AB} and produces an estimator $\rho_{AB}^{(n)}$ such that

$$\mathbb{E}_{\rho_{AB}^{(n)}} \left(1 - F(\rho_{AB}^{(n)}, \rho_{AB}) \right) \leq \frac{C}{n}\,, \tag{15}$$

where C is a universal constant.

ALERT we are estimating with respect to the *infidelity*.

B obtains the classical estimate $\rho_{AB}^{(n)}$. Then, they send it to A.

R. O'Donnell and J. Wright, arXiv:1612.00034 (2016).

Distill with bounded error

Now, what is the error on the remaining copies? After many calculations, we prove that is bounded by

$$\mathbb{E}_{\rho_{AB}^{(n)}} \frac{1}{2} \left\| \rho_{AB}^{\otimes n} - \rho_{AB}^{(n)\otimes n} \right\|_1 \ . \tag{16}$$

We use the *multiplicativity* of infidelity and the Fuchs–van de Graaf inequalities to obtain:

$$\mathbb{E}_{\rho_{AB}^{(n)}} \frac{1}{2} \left\| \rho_{AB}^{\otimes n} - \rho_{AB}^{(n)\otimes n} \right\|_1 \leq \sqrt{1 - \left(1 - \frac{C}{n}\right)^{2n}} \to \sqrt{1 - e^{-2C}} < 1 \ . \tag{17}$$

Distill with bounded error

Then, we apply the optimal LOCC to distill entanglement for the estimated state $\rho_{AB}^{(n)}$ on the remaining copies of the true state ρ_{AB}. Finally, we obtain a number of Bell states with a bounded error. If we lower the error, we increase the rate and vice versa.

$$\hat{E}_d^{\dagger}(\mathcal{X}) \geq \inf_{\rho_{AB} \in \mathcal{X}} E_d(\rho_{AB}) \,. \tag{18}$$

Conclusions and perspectives

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- A universal protocol to distill entanglement
- Can we find a zero-error universal protocol?
- Can we apply this idea to other quantum resources? (Talk by Kaito)

Thank you!

Keyl algorithm

Use the fact that $\rho_{AB}^{\otimes n}$ is permutationally invariant. Then, measure the the state on the Young frames.

M. Keyl and R. Werner, Phys. Rev. A **64**, 052311 (2001)