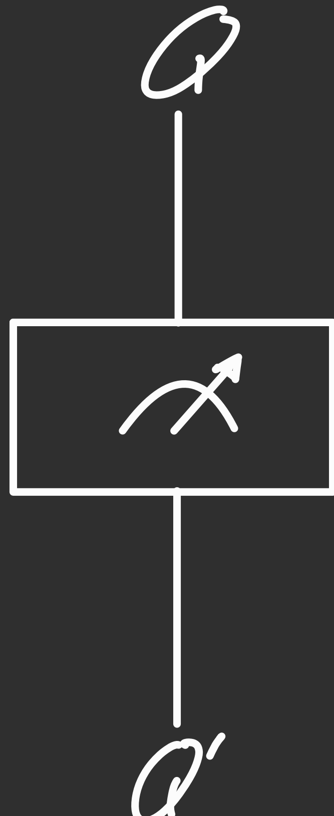
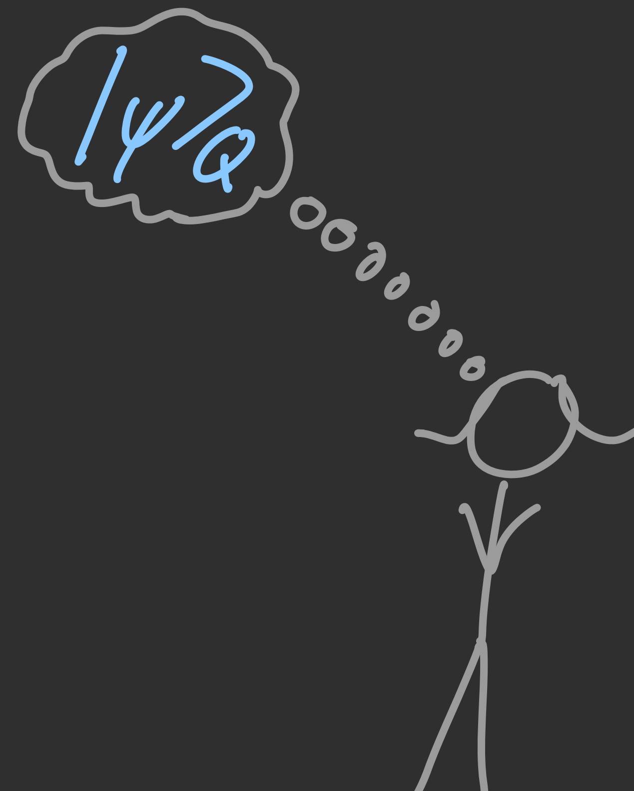
Why robustness matters in quantum state representations

Renato Renner
joint work with Ladina Hausmann

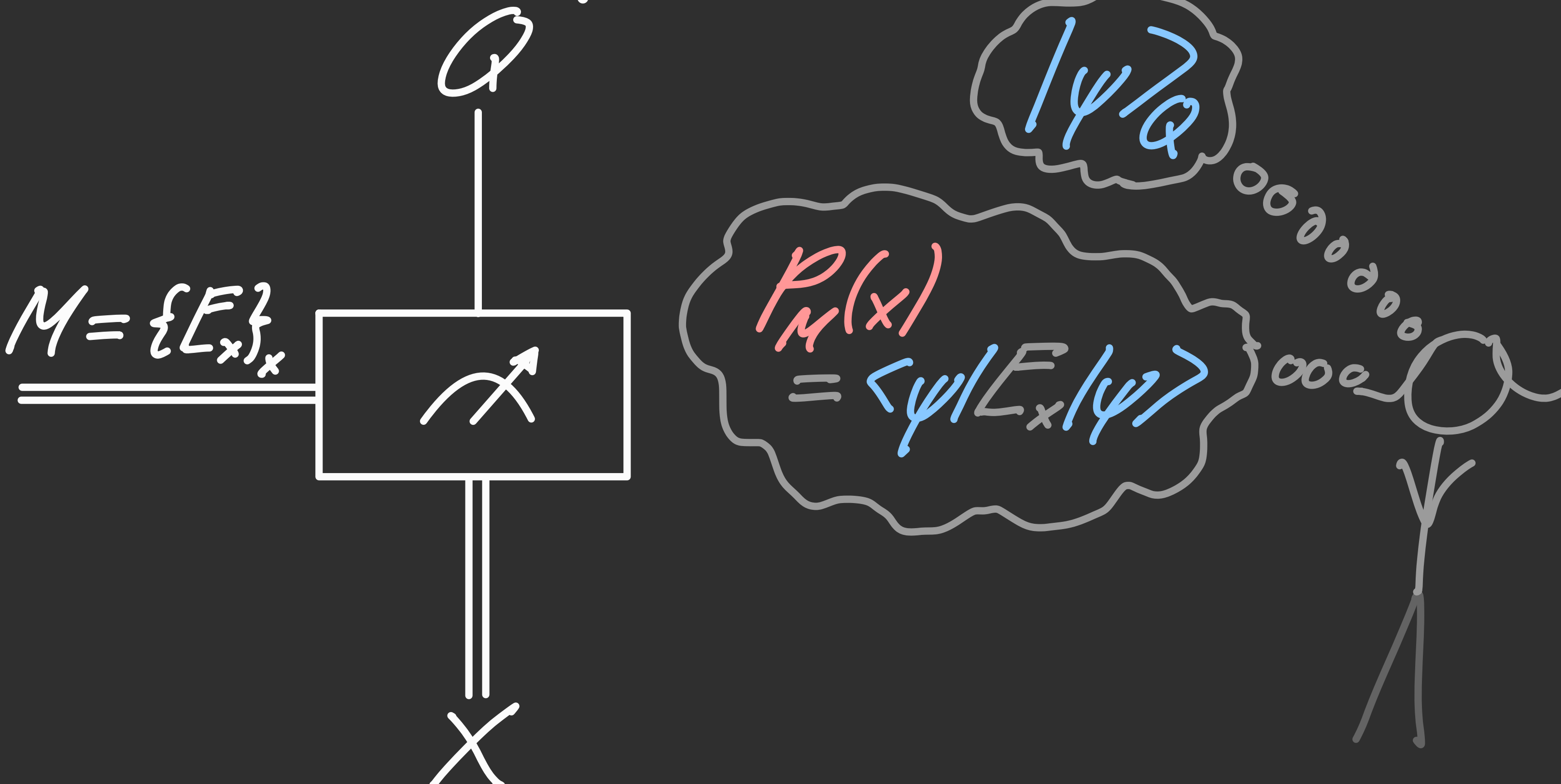
based on arXiv:2601.18872

What is a quantum state?

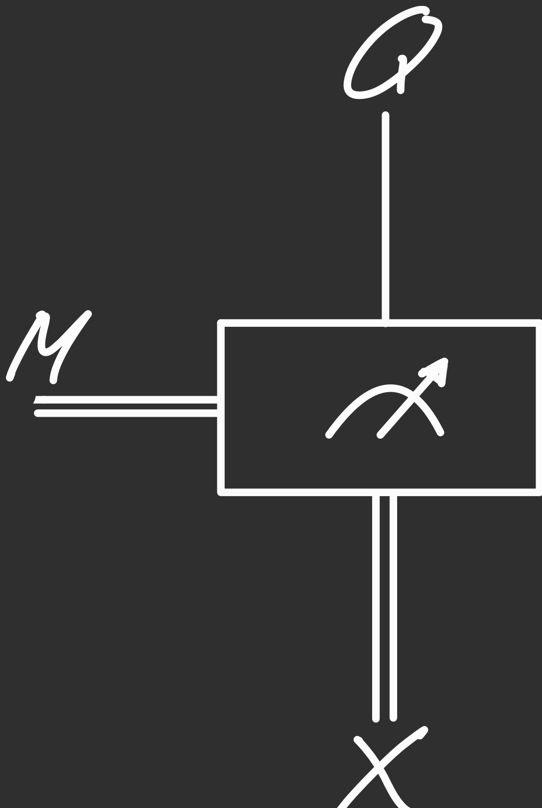




What is a quantum state?

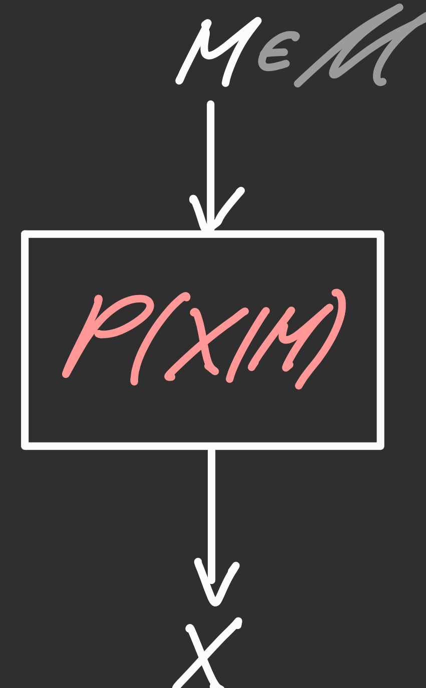


Just a representation of probabilities?



Generalised Probabilistic Theories (GPTs)

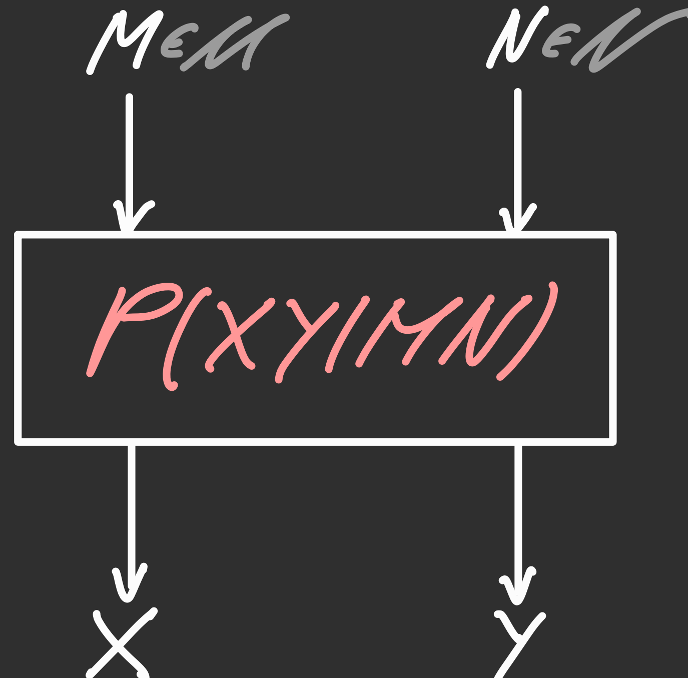
$|\psi\rangle_Q \;\hat{=}$



Holds if $\mathcal{M}$ is tomographically complete.

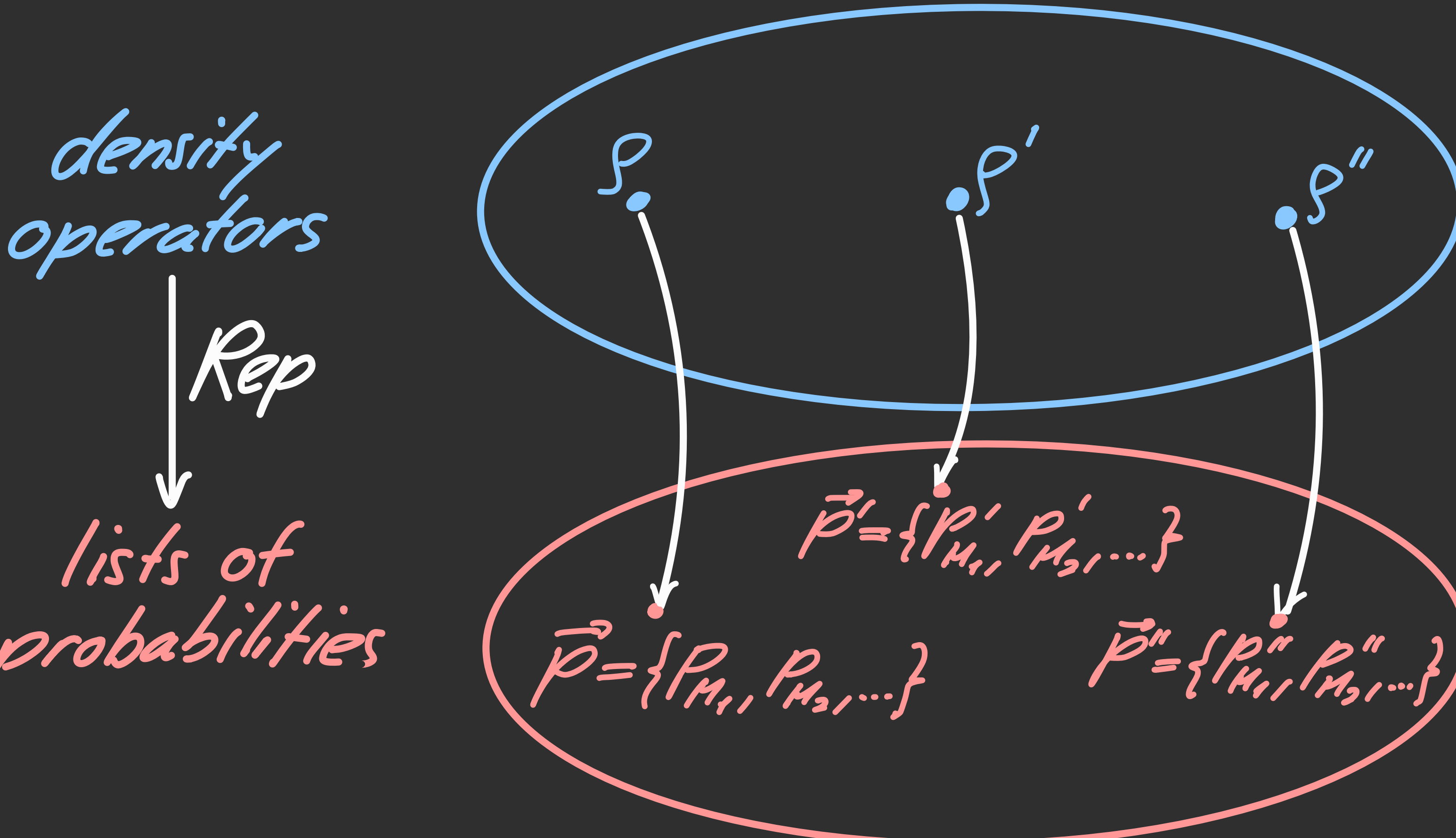
Generalised Probabilistic Theories (GPTs)

$$\rho_{AB} \;\hat{=}$$



Holds if $\mathcal{M} \times \mathcal{N}$ is tomographically complete.

Probability representation



Probability representations as a tool

$$
\begin{tikzcd}
\rho \arrow[d, "\mathrm{Rep}"] & & \text{property of } \rho \\
\vec{p} \arrow[rr, Rightarrow, "\text{classical}", "\text{methods}"'] & & \text{property of } \vec{p} \arrow[u, "\mathrm{Rep}^{-1}"']
\end{tikzcd}
$$

Example 1

Theorem (de Finetti)

$P_{X_1 \ldots X_n}$ exchangeable

$\Longleftrightarrow \forall N > n:$

$\exists$ permutation-invariant $P_{X_1 \ldots X_n X_{n+1} \ldots X_N}$

$\Longrightarrow$

$P_{X_1 \ldots X_n} \in \mathrm{conv}(Q_X^{\times n})$

$\Longleftrightarrow P_{X_1 \ldots X_n} = \int dQ_X \, Q_X^{\times n}$ [de Finetti; 1937]

Example 1

$$\begin{tikzcd}
\rho_{A_1 \cdots A_n} \text{ exchangable} \arrow[d, Rightarrow] & \rho_{A_1 \cdots A_n} \in \mathrm{conv}\left(\sigma_A^{\otimes n}\right) \\
P_{X_1 \cdots X_n} \text{ exchangable} \arrow[r, Rightarrow, "\text{de Finetti}"] & P_{X_1 \cdots X_n} \in \mathrm{conv}\left(Q_X^{\times n}\right) \arrow[u, Rightarrow]
\end{tikzcd}$$

Example 1

$$
\begin{tikzcd}
\rho_{A_1\cdots A_n} \text{ exchangable} \arrow[r, Rightarrow, "\text{quantum}", "\text{de Finetti}"'] \arrow[d, Rightarrow] & \rho_{A_1\cdots A_n} \in \mathrm{conv}\left(\sigma_A^{\otimes n}\right) \\
P_{X_1\cdots X_n} \text{ exchangable} \arrow[r, Rightarrow, "\text{classical}", "\text{de Finetti}"'] & P_{X_1\cdots X_n} \in \mathrm{conv}\left(Q_X^{\times n}\right) \arrow[u, Rightarrow]
\end{tikzcd}
$$

[Caves, Fuchs, Schack, 2002]

Does this work in general?

$$\begin{tikzcd}
\rho \arrow[d, "\mathrm{Rep}"] & & \text{property of } \rho \\
\vec{p} \arrow[r, "\text{classical}", "\text{methods}"'] & \text{property of } \vec{p} \arrow[ur, "\mathrm{Rep}^{-1}"'] &
\end{tikzcd}$$

Example 2

Theorem (approximate de Finetti)

$P_{X_1 \cdots X_N Z}$ permutation invariant (relative to Z)

$\Longrightarrow$

$$P_{X_1 \cdots X_n Z} \xrightarrow{N \to \infty} \mathrm{conv}(Q_X^{\times n} \times Q_Z)$$

[Diaconis, Freedman, 1980]

Example 2 (?)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c} \rho_{A_1 \cdots A_N E} \\ \text{permutation} \\ \text{invariant} \end{array} \arrow[d, Rightarrow] & \rho_{A_1 \cdots A_n E} \xrightarrow{N \to \infty} \mathrm{conv}\left(\sigma_A^{\otimes n} \otimes \sigma_E\right) \\
\begin{array}{c} P_{X_1 \cdots X_N Z} \\ \text{permutation} \\ \text{invariant} \end{array} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & P_{X_1 \cdots X_n Z} \xrightarrow{N \to \infty} \mathrm{conv}\left(Q_X^{\times n} \times Q_Z\right) \arrow[u, Rightarrow]
\end{tikzcd}
$$

Example 2 $\lightning$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c} \rho_{A_1 \cdots A_N E} \\ \text{permutation} \\ \text{invariant} \end{array} \arrow[r, Rightarrow, "/" marking] \arrow[d, Rightarrow] & \begin{array}{c} \rho_{A_1 \cdots A_n E} \\ \xrightarrow{N \to \infty} \mathrm{conv}(\sigma_A^{\otimes n} \otimes \sigma_E) \end{array} \\
\begin{array}{c} P_{X_1 \cdots X_N Z} \\ \text{permutation} \\ \text{invariant} \end{array} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & \begin{array}{c} P_{X_1 \cdots X_n Z} \\ \xrightarrow{N \to \infty} \mathrm{conv}(Q_X^{\times n} \times Q_Z) \end{array} \arrow[u, Rightarrow]
\end{tikzcd}
$$

Example 2 ↯

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c} \rho_{A_1\cdots A_N E} \\ \text{permutation invariant} \end{array} \arrow[r, Rightarrow, "/" marking] \arrow[d, Rightarrow] & \rho_{A_1\cdots A_n E} \xrightarrow{N\to\infty} \mathrm{conv}(\sigma_A^{\otimes n}\otimes\sigma_E) \\
\begin{array}{c} P_{X_1\cdots X_N Z} \\ \text{permutation invariant} \end{array} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & P_{X_1\cdots X_n Z} \xrightarrow{N\to\infty} \mathrm{conv}(Q_X^{\times n}\times Q_Z) \arrow[u, Rightarrow, "/" marking]
\end{tikzcd}
$$

Remark on "system size"

$\rho_A^{(1)}$ $\quad$ rank $\in \mathbb{N}$

$\rho_A^{(2)}$ $\quad$ rank $\in \mathbb{N}$

$\rho_A^{(3)}$ $\quad$ rank $\in \mathbb{N}$

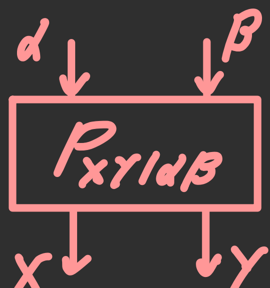
$\vdots$ $\quad$ $\sup(\text{rank}) \notin \mathbb{N}$

Example 3: Device-idependent QKD

ρ_{AB}

strongly entangled

$\Downarrow$

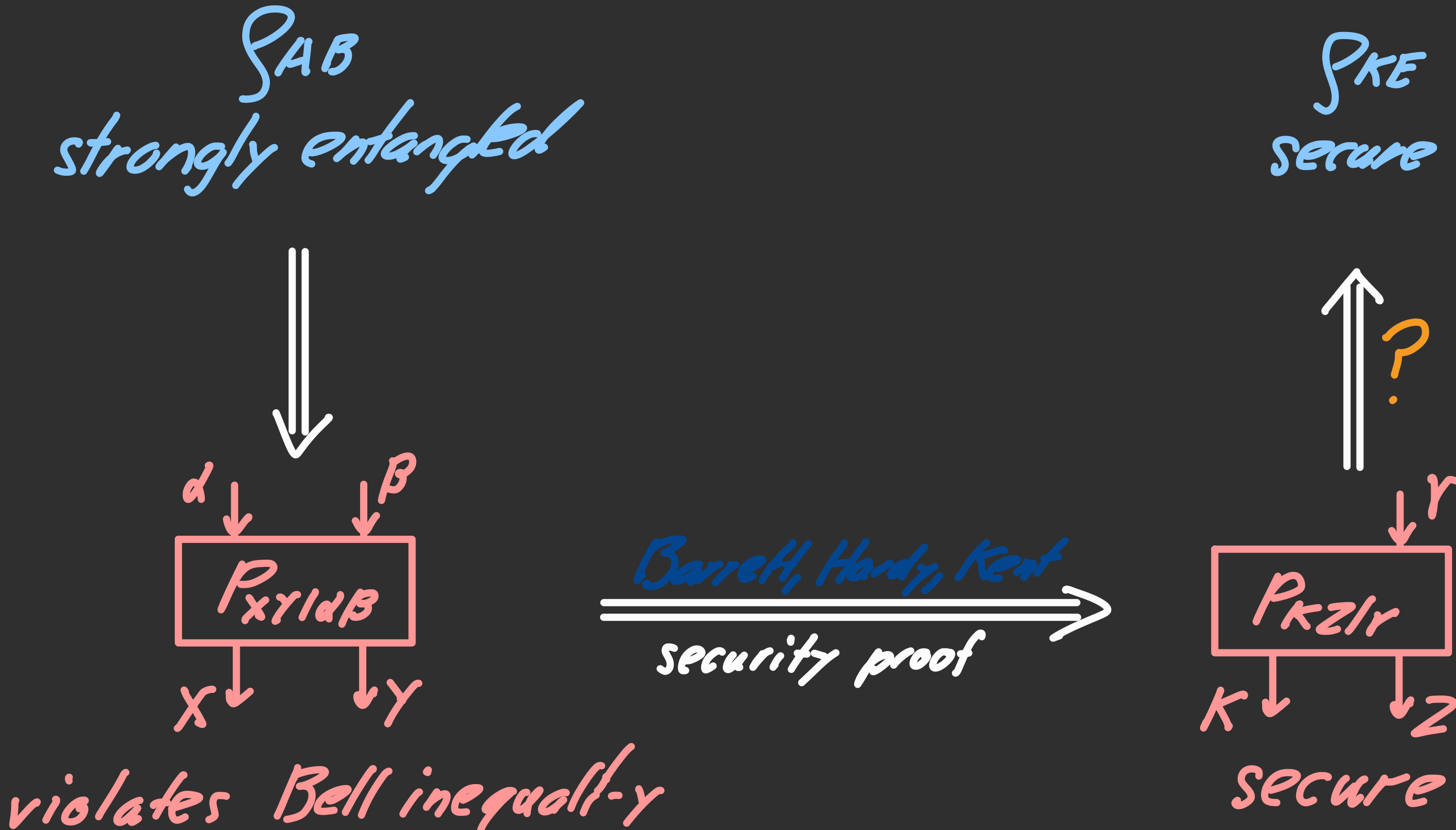


violates Bell inequality

quantum state

GPT state

Example 3: Device-idependent QKD



Identifying the problem

Example 1

$\rho_{A_1 \cdots A_n}$ exchangable

$\Downarrow$

$P_{X_1 \cdots X_n}$ exchangable $\xRightarrow{\text{de Finetti}}$ $P_{X_1 \cdots X_n} \in \mathrm{conv}(Q_X^{\times n})$

$\Uparrow$ ✓

$\rho_{A_1 \cdots A_n} \in \mathrm{conv}(\sigma_A^{\otimes n})$

Example 2

$\rho_{A_1 \cdots A_N E}$ permutation invariant

$\Downarrow$

$P_{X_1 \cdots X_N Z}$ permutation invariant $\xRightarrow[\text{de Finetti}]{\text{approximate}}$ $P_{X_1 \cdots X_n Z} \xrightarrow{N \to \infty} \mathrm{conv}(Q_X^{\times n} \times Q_Z)$

$\not\Uparrow$

$\rho_{A_1 \cdots A_n E} \xrightarrow{N \to \infty} \mathrm{conv}(\sigma_A^{\otimes n} \otimes \sigma_E)$

Identifying the problem

Example 1

$$
\begin{tikzcd}
\underset{\text{exchangable}}{\rho_{A_1\cdots A_n}} \arrow[d, Rightarrow] & \rho_{A_1\cdots A_n} \in \mathrm{conv}(\sigma_A^{\otimes n}) \\
\underset{\text{exchangable}}{P_{X_1\cdots X_n}} \arrow[r, Rightarrow, "\text{de Finetti}"] & P_{X_1\cdots X_n} \in \mathrm{conv}(Q_X^{\times n}) \arrow[u, Rightarrow, "\checkmark"']
\end{tikzcd}
$$

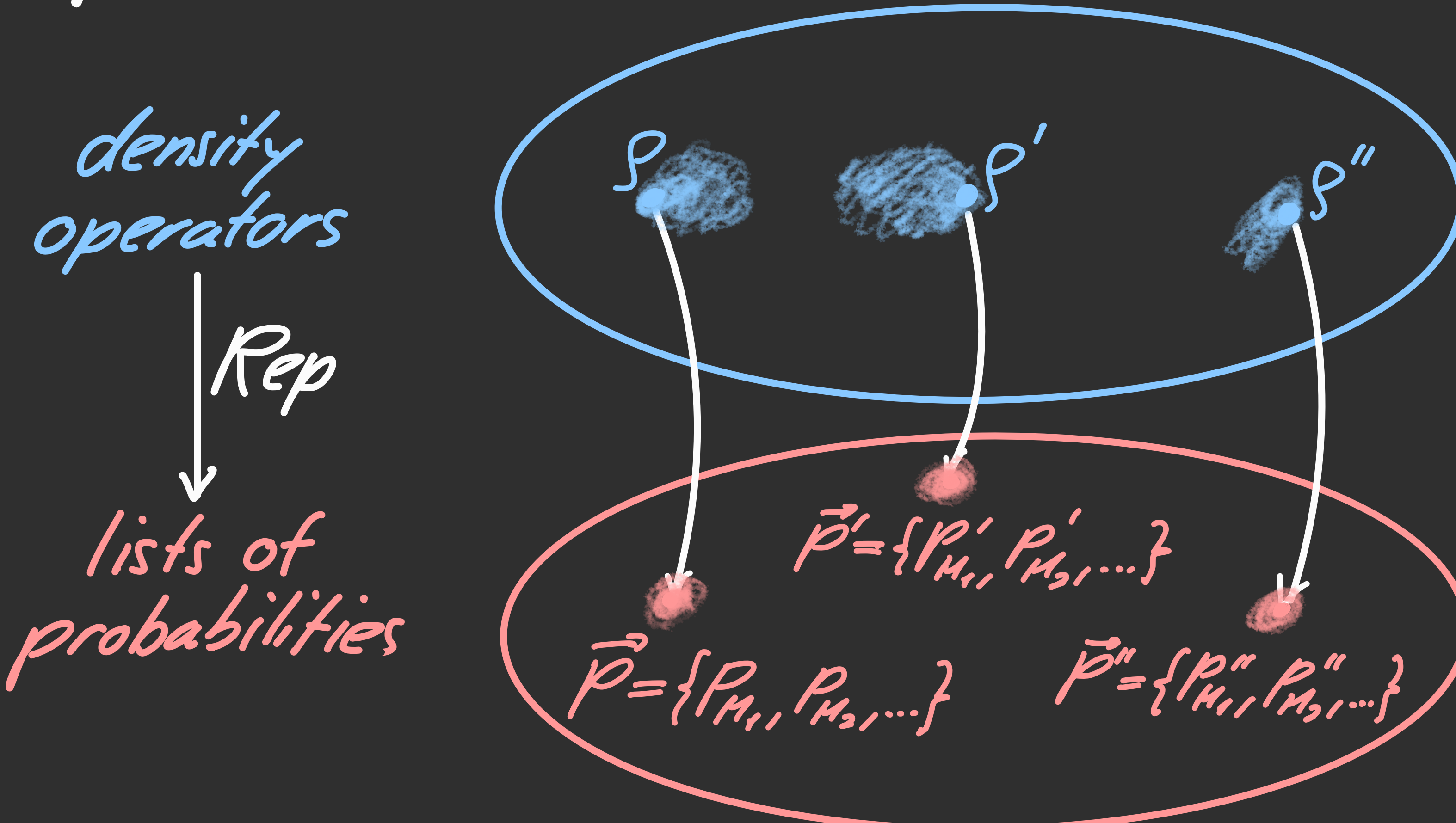
exact statement

Example 2

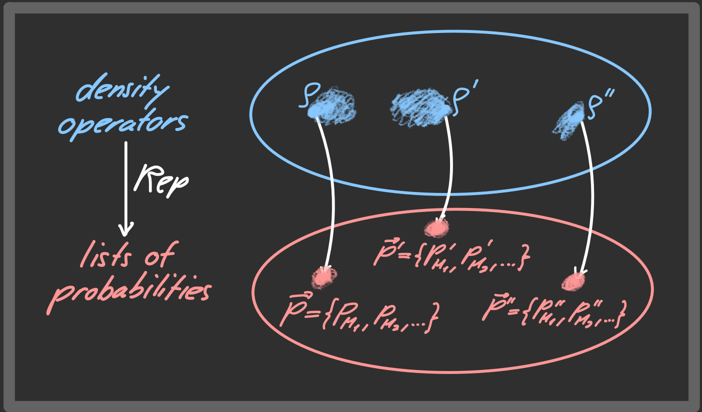
$$
\begin{tikzcd}
\underset{\text{permutation invariant}}{\rho_{A_1\cdots A_N E}} \arrow[d, Rightarrow] & \rho_{A_1\cdots A_n E} \xrightarrow{N\to\infty} \mathrm{conv}(\sigma_A^{\otimes n}\otimes\sigma_E) \\
\underset{\text{permutation invariant}}{P_{X_1\cdots X_N Z}} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & P_{X_1\cdots X_n Z} \xrightarrow{N\to\infty} \mathrm{conv}(Q_X^{\times n}\times Q_Z) \arrow[u, Rightarrow, "/"']
\end{tikzcd}
$$

approximate statement

Requirement: "Robustness"



Result 0



Proposition

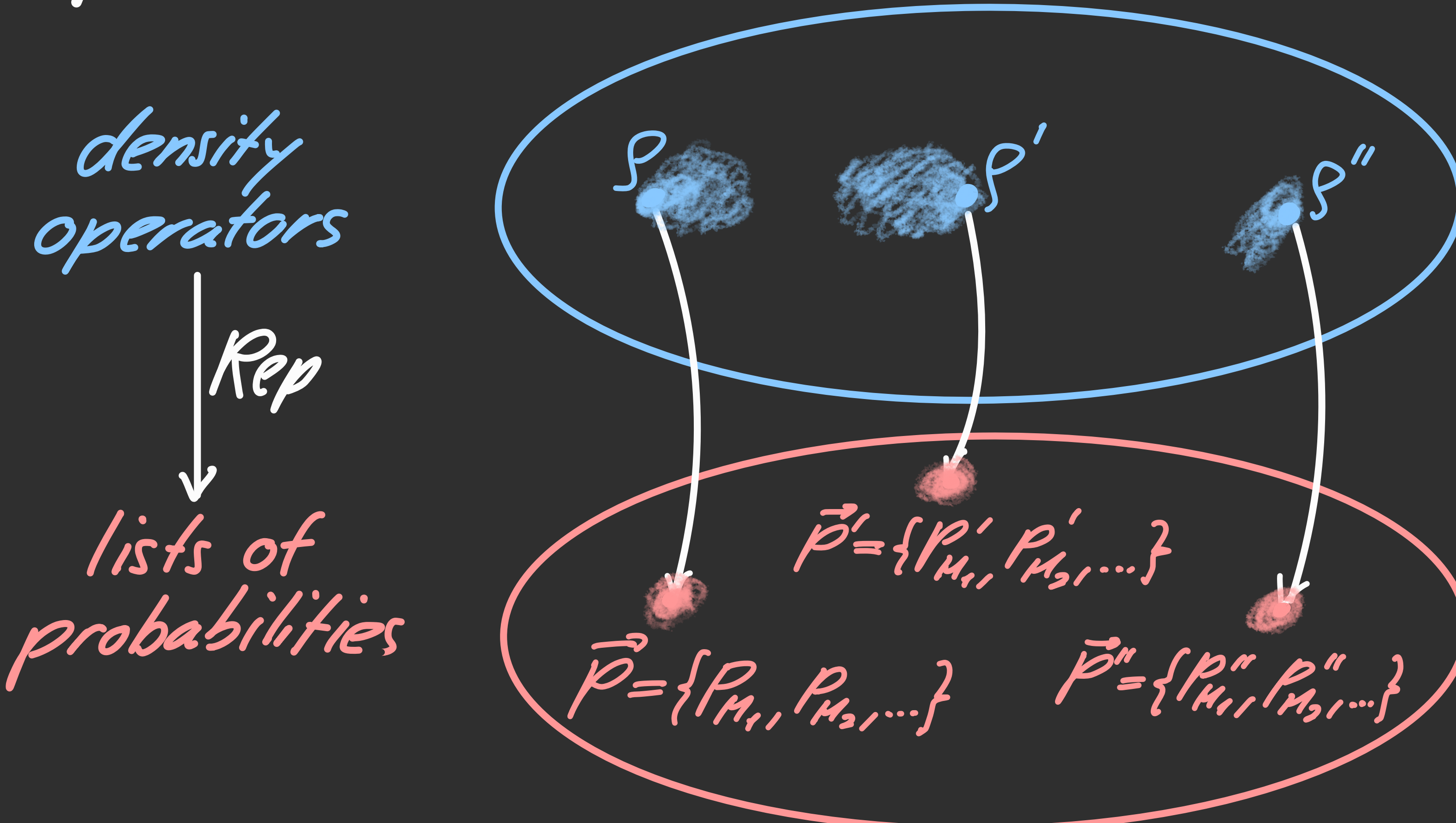
The representation map Rep and its inverse are continuous.*

* except for pathological measurement sets

Continuity is not enough

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\rho_{A_1\cdots A_N E}\\ \text{permutation invariant}\end{array} \arrow[r, Rightarrow, "/" marking] \arrow[d, "\text{Rep}"', "\text{continuous}"] & \begin{array}{c}\rho_{A_1\cdots A_n E}\\ \xrightarrow{N\to\infty} \mathrm{conv}(\sigma_A^{\otimes n}\otimes\sigma_E)\end{array} \\
\begin{array}{c}P_{X_1\cdots X_N Z}\\ \text{permutation invariant}\end{array} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & \begin{array}{c}P_{X_1\cdots X_n Z}\\ \xrightarrow{N\to\infty} \mathrm{conv}(Q_X^{\times n}\times Q_Z)\end{array} \arrow[u, "\text{Rep}^{-1}", "\text{continuous}"']
\end{tikzcd}
$$

Requirement: "Robustness"



Result 1

Theorem

The diagram commutes if Rep^{-1} is continuous on $\mathrm{span}(\mathrm{Im}(\vec{P}))$

$$\begin{tikzcd}
\rho \arrow[rr, Rightarrow] \arrow[d, "\mathrm{Rep}"] & & \text{property of } \rho \\
\vec{p} \arrow[rr, Rightarrow] & & \text{property of } \vec{P} \arrow[u, "\mathrm{Rep}^{-1}"']
\end{tikzcd}$$

Result 1

Theorem

The diagram commutes if Rep^{-1} is continuous on $\mathrm{span}(\mathrm{Im}(\mathrm{Rep}))$

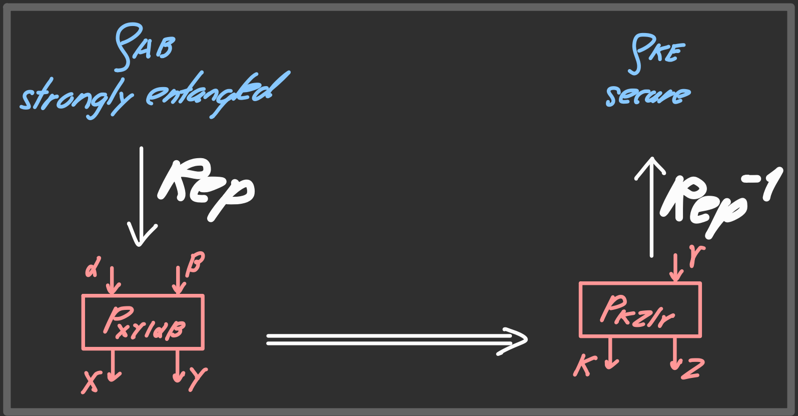
$$
\begin{tikzcd}
\rho \arrow[r, Rightarrow] \arrow[d, "\mathrm{Rep}"] & \text{property of } \rho \\
\vec{p} \arrow[r, Rightarrow] & \text{property of } \vec{p} \arrow[u, "\mathrm{Rep}^{-1}"']
\end{tikzcd}
$$

If this holds, we call Rep "robust."

Result 2

__Theorem__ (informal)

If Rep conserves structure❄ then it is not robust.



❄ ⟶ poster by Ladina Hausmann

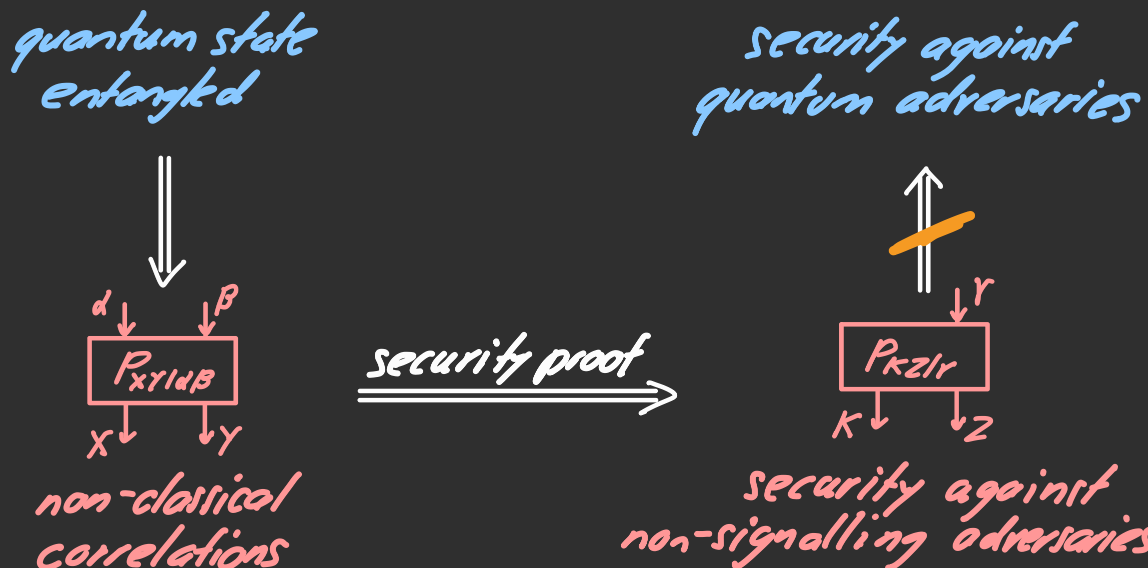
Conclusion

Probability representations do not faithfully represent quantum states!

$$
\begin{tikzcd}
\text{quantum state permutation invariant} \arrow[d, Rightarrow] & \text{quantum state in conv(IID)} \\
\text{probability dist. permutation invariant} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & \text{probability dist. in conv(IID)} \arrow[u, Rightarrow, "/" description]
\end{tikzcd}
$$

Implication I

→ Security proofs in probability representations do not apply to quantum theory.



Implication II

→ de Finetti-type theorems do not generally hold in general theories

$$
\begin{tikzcd}
\text{generalised state permutation invariant} \arrow[d, Rightarrow] & \text{generalised state in conv(IID)} \\
\text{probability dist. permutation invariant} \arrow[r, Rightarrow, "\text{approximate}", "\text{de Finetti}"'] & \text{probability dist. in conv(IID)} \arrow[u, Rightarrow, "/" marking]
\end{tikzcd}
$$

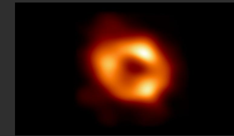
☀ see Giulia Mazzola's talk for operational significance

Implication III

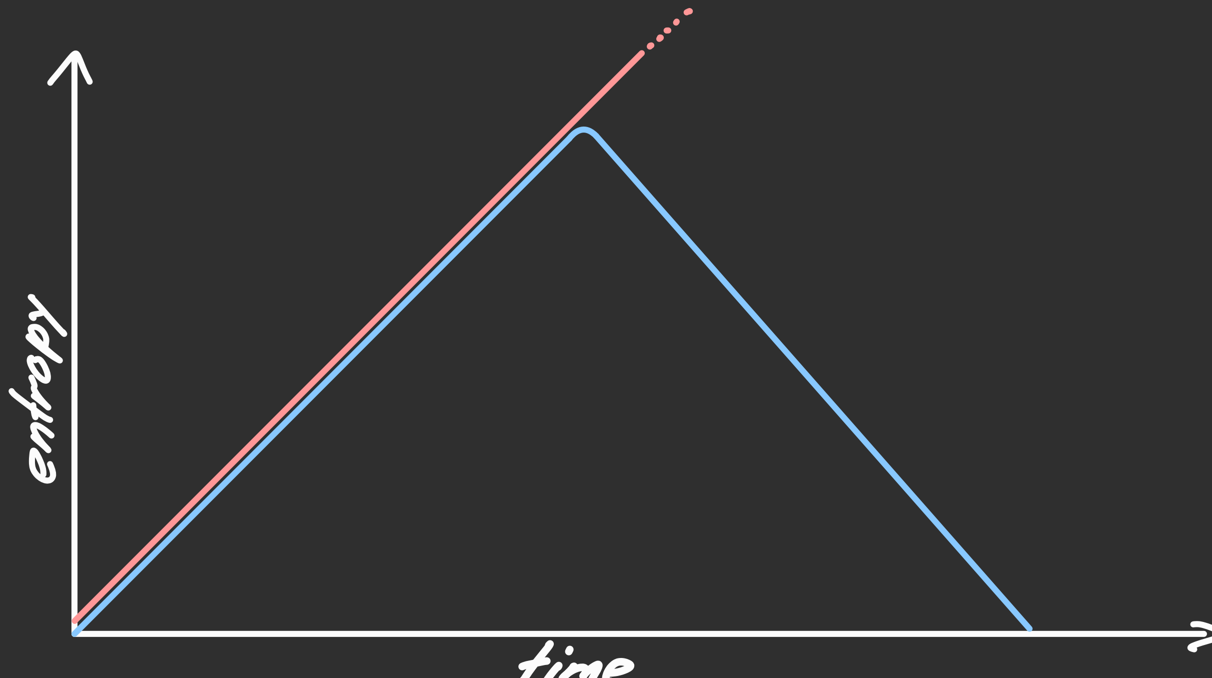
→ correlation fuctions are not faithful representations of quantum states

$$\begin{tikzcd}
\text{QFT state} \arrow[d, Rightarrow] & & \text{property of QFT state} \\
\text{correlation functions} \arrow[rr, Rightarrow, "\text{approximate}", "\text{argument}"'] & & \text{property of representation} \arrow[u, Rightarrow, "/" marking]
\end{tikzcd}$$

Implication IV



→ Path integral approach cannot consistently inform us about Hawking radiation




$|\psi\rangle_Q \neq$
$P_{M_1}(x_1), P_{M_1}(x_2), \ldots$
$P_{M_2}(x_1), P_{M_2}(x_2), \ldots$
$P_{M_3}(x_1), P_{M_3}(x_2), \ldots$

Thanks for your attention

arXiv:2601.18872