Optimising quantum data hiding

Joint work with Ludovico Lami

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Mele and Lami, *Optimising quantum data hiding,* arXiv:2510.03538

Quantum data hiding

Definition (informal)

ρ_{AB} and σ_{AB} form a pair of **quantum data hiding** states if:

- They are perfectly distinguishable via global operations;
- They are nearly indistinguishable via LOCC.

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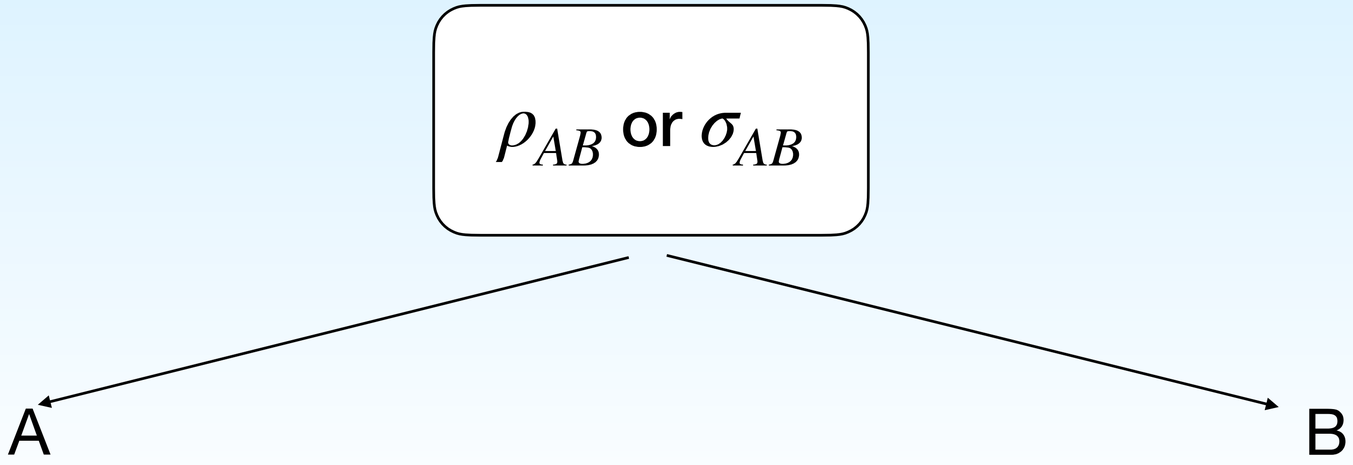

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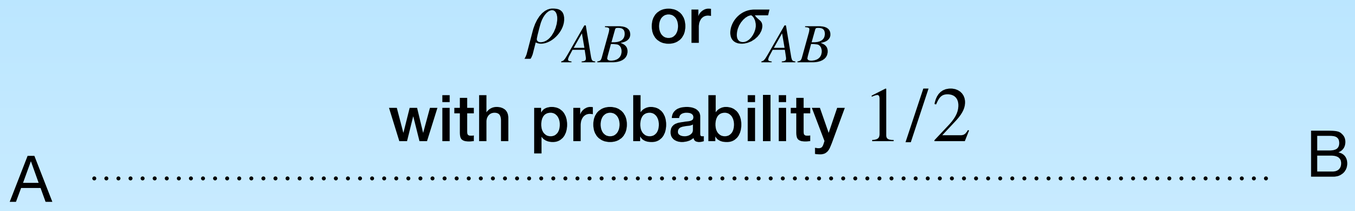
Motivations:

- Cryptography
- Bound entanglement

Horodecki$^{\otimes 3}$, Oppenheim, *PRL* (2005)
Christandl and Ferrara, *PRL* (2017)

LOCC norm

Matthews, Wehner, and Winter; *CMP* (2009)



- What is the best probability of successfully discriminating via **global operations**?

$$P_{\text{succ}}(\rho_{AB}, \sigma_{AB}) = \frac{1}{2}\left(1 + \frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_1\right)$$

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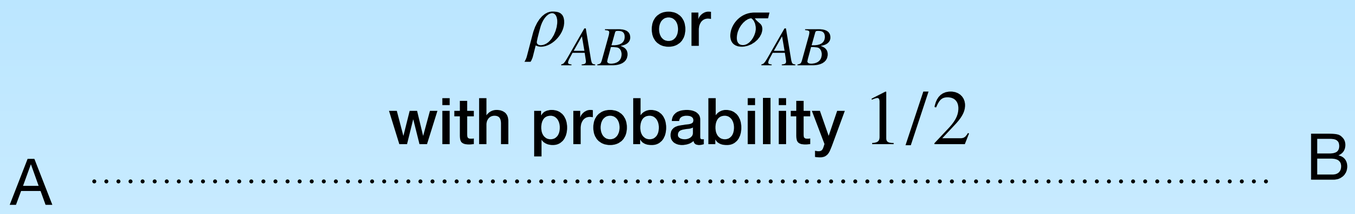

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$$P_{\text{succ}}^{(\text{LOCC})}(\rho_{AB}, \sigma_{AB}) = \frac{1}{2}\left(1 + \frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_{\text{LOCC}}\right)$$

(This can be regarded as the definition of LOCC norm)

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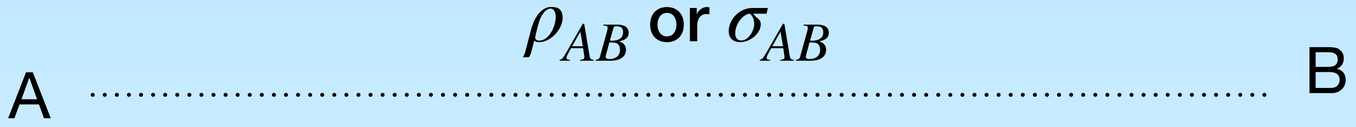
[Eggeling and Werner; PRL (2002)] exhibit **separable** states ρ_{AB} and σ_{AB} such that:

$\frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_{\mathrm{LOCC}} \leq \varepsilon$ but $\frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_{1} \neq 1$ (i.e., they are not *perfectly* orthogonal)

Does there exist a pair of **separable** ε-quantum data hiding states?

(i.e. a pair of states that are separable, orthogonal, and ε-close in LOCC-norm)



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→ Irreversibility of hiding secrets into quantum states via LOCC

Theorem 1 (Existence of separable ε-quantum data hiding states)

For all $\varepsilon \in (0,1)$ there exist **separable** states ρ_{AB} and σ_{AB} on $\mathbb{C}^D \otimes \mathbb{C}^D$ such that:

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- All ε-quantum data hiding states on $\mathbb{C}^D \otimes \mathbb{C}^D$ must satisfy $D = \Omega(1/\varepsilon)$.

Matthews, Wehner, and Winter; *CMP* (2009)
Lami, Palazuelos, and Winter; *CMP* (2018)

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$$\frac{1}{2}\|\rho_1^{(k)} - \rho_0^{(k)}\|_1 = \left(\frac{1}{2}\|\sigma_1 - \sigma_0\|_1\right)^k \xrightarrow{k\to\infty} \begin{cases} 0 & \text{if } \frac{1}{2}\|\sigma_1 - \sigma_0\|_1 < 1 \\ 1 & \text{otherwise} \end{cases}$$

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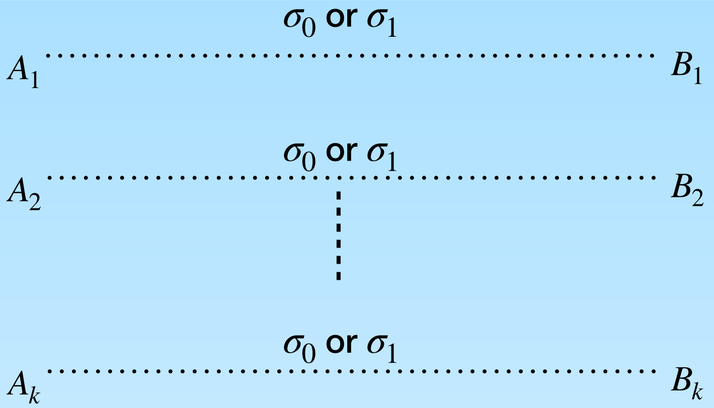

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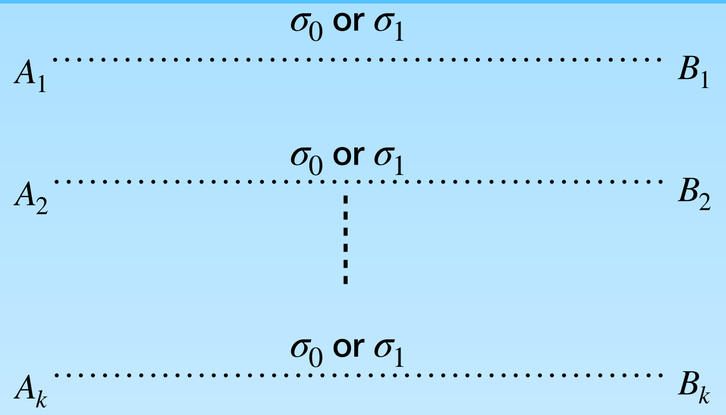

Open problem (Sub-multiplicativity of LOCC-norm)

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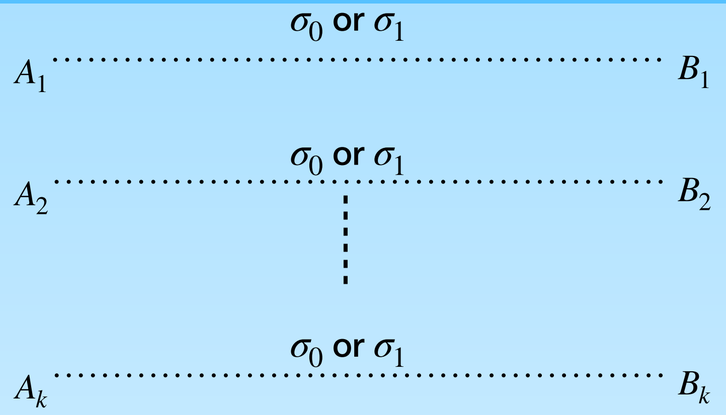

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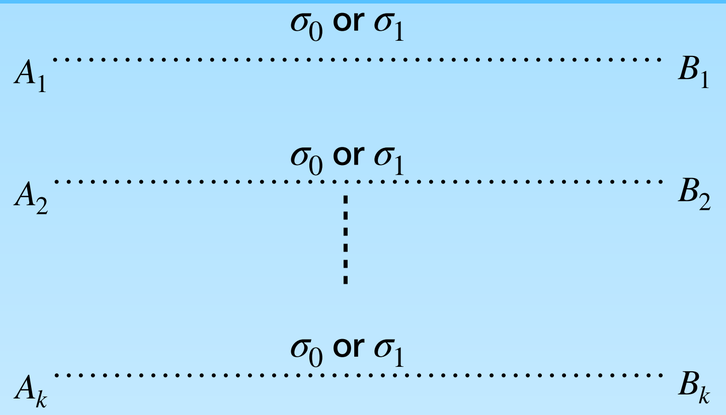

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Proof idea of Theorem 1 (Existence of separable ε-quantum data hiding states):

Pick σ_0, σ_1 separable-orthogonal states satisfying $\frac{1}{2}\|\sigma_1 - \sigma_0\|_{\mathrm{LOCC}} < 1$. Then, use *parity trick*.

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- 1st attempt: $(U \otimes U)\,\rho\,(U \otimes U)^\dagger = \rho \quad \forall U$ ⟶ No examples of separable orthogonal states

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- 3rd attempt: $(U \otimes U)\,\rho\,(U \otimes U)^\dagger = \rho \quad \forall U \in$ *hyperoctahedral* group → It works!

Twirling w.r.t. hyperoctahedral group (permutations+signs)

$$\frac{1}{|G|}\sum_{U\in G}(U\otimes U)\,X\,(U\otimes U)^\dagger=\sum_{i=0}^{3}\mathrm{Tr}[X\Theta_i]\,\frac{\Theta_i}{\mathrm{Tr}[\Theta_i]}$$

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$\Theta_0, \Theta_1, \Theta_2, \Theta_3$ are mutually orthogonal projectors, and satisfy

$$\sum_{i=0}^{3}\Theta_i = \mathbb{I}$$

$$\sigma_1 := \frac{1}{2}\left(\frac{\Theta_1}{\mathrm{Tr}\Theta_1} + \frac{\Theta_3}{\mathrm{Tr}\Theta_3}\right)$$ “Biblical state”

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“Biblical state”

σ_0 and σ_1 are separable and orthogonal

Even state: $\rho_0^{(k)} := \frac{1}{2^{k-1}} \sum_{\substack{x_1,\ldots,x_k \in \{0,1\} \\ x_1+\cdots+x_k \equiv 0 \pmod 2}} \sigma_{x_1} \otimes \cdots \otimes \sigma_{x_k}$

Odd state: $\rho_1^{(k)} := \frac{1}{2^{k-1}} \sum_{\substack{x_1,\ldots,x_k \in \{0,1\} \\ x_1+\cdots+x_k \equiv 1 \pmod 2}} \sigma_{x_1} \otimes \cdots \otimes \sigma_{x_k}$

$$\frac{1}{2}\|\rho_0^{(k)} - \rho_1^{(k)}\|_{\mathrm{PPT}} \overset{\text{Symmetry}}{=} \text{linear program}$$

PPT norm between the even and odd states

$$\frac{1}{2}\|\rho_0^{(k)} - \rho_1^{(k)}\|_{\mathrm{PPT}} = \inf_{\mathbf{x}\in\mathbb{R}^{4^k}} \left[\|\mathbf{x}\|_1 + \|W^{\otimes k}\mathbf{x} - \mathbf{r}^{\otimes k}\|_1\right]$$

where $W \in \mathbb{R}^{4\times 4}$ and $\mathbf{r} \in \mathbb{R}^4$.

$$\mathbf{r} := \begin{pmatrix} \frac{1}{2d} \\ -\frac{1}{4} \\ \frac{d-1}{2d} \\ -\frac{1}{4} \end{pmatrix} \qquad W := \begin{pmatrix} \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & -\frac{1}{d} \\ 1-\frac{1}{d} & 1-\frac{1}{d} & -\frac{1}{d} & \frac{1}{d} \\ \frac{d-1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{d-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- $$\frac{1}{2}\|\rho_0^{(k)} - \rho_1^{(k)}\|_{\mathrm{PPT}} = \inf_{\mathbf{x}\in\mathbb{R}^{4^k}} \left[\|W^{\otimes k}\mathbf{x} - \mathbf{r}^{\otimes k}\|_1 + \|\mathbf{x}\|_1\right]$$

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Tikhonov-regularised least squares

$$\inf_{\mathbf{x}} \left[\|A\mathbf{x} - \mathbf{b}\|_2^2 + \|\mathbf{x}\|_2^2\right] = \sum_i \frac{(\mathbf{u}_i^\intercal \mathbf{b})^2}{1+\sigma_i^2}$$

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- The last step of the proof uses the Sanov's theorem

(Sanov's theorem quantifies how unlikely it is that an empirical distribution of k i.i.d. samples deviates from the true distribution)

Proposition (Upper bound on the LOCC-norm between the even and odd states)

The even and odd states satisfies

$$\frac{1}{4}\|\rho_0^{(k)} - \rho_1^{(k)}\|_{\mathrm{LOCC}} \leq \mu^k$$

where $\mu \in (0,1)$.

$$\mu = \sqrt{1 - \frac{\frac{5}{8} + \frac{1}{d}\left(\frac{1}{4} + \frac{2}{d} + \frac{9}{d^2} - \frac{6}{d^3} - \sqrt{2}\left(\frac{9}{4} + \frac{3}{d} + \frac{1}{d^2}\right)\sqrt{1 - \frac{2}{d + \frac{4}{d}}}\right)}{1 + \frac{2}{d} + \frac{4}{d^2}}}$$

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For k sufficiently large, it holds $\frac{1}{2}\|\rho_0^{(k)} - \rho_1^{(k)}\|_{\mathrm{LOCC}} \leq \varepsilon$, i.e.

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All ε-quantum data hiding states on $\mathbb{C}^D \otimes \mathbb{C}^D$ must satisfy $D = \Omega(1/\varepsilon)$. Matthews, Wehner, and Winter; *CMP* (2009)
Lami, Palazuelos, and Winter; *CMP* (2018)

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This would imply:

Conjecture

Let σ_0, σ_1 separable-orthogonal states satisfying $\frac{1}{2}\|\sigma_1 - \sigma_0\|_{\mathrm{LOCC}} < 1$. Then, for k sufficiently large, the associated even and odd states form a pair of **separable** ε-quantum data hiding states.

Thank you!