Almost-iid information theory

operational state transformation rates

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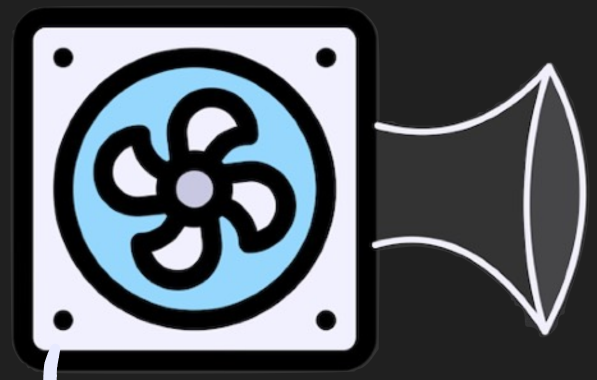
Motivation



source

ρ

Density matrix:

- $\rho^\dagger = \rho$ (hermit.)
- $\rho \geq 0$ (pos. def.)
- $\mathrm{tr}(\rho) = 1$ (norm.)

Measurements (POVM $\{M_x\}_{x\in\mathcal{X}}$)

$$p_x = \mathrm{tr}(M_x \rho) = \mathbb{E}(M_x)$$

↳ probabilities

Entropy (v.N.)

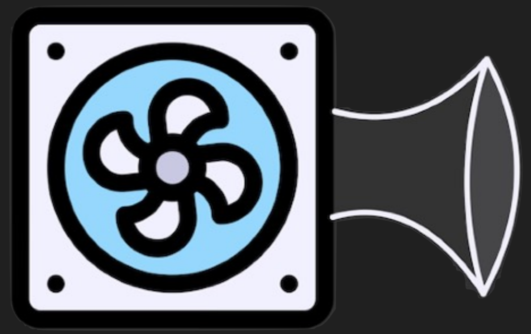
$$H(\rho) = -\mathrm{tr}(\rho \log(\rho)) = \mathbb{E}[-\log(\rho)]$$

Entanglement measures & rates

E_C, E_D, … (entanglement cost, distillation, …)

…

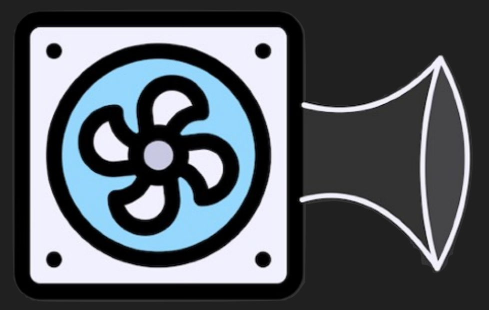
$\Rightarrow$ Operationally, such quantities are inferred from many copies of ρ.



ρ ρ ρ … ρ

Motivation

common assumption

 $\overbrace{\rho \quad \rho \quad \rho \ \cdots \ \rho}^{\varrho^{(n)}} \longrightarrow \quad \rho^{(n)} \overset{?}{=} \boxed{\rho^{\otimes n}}$ (iid)

Example: How to determine $P_X(x) = \mathrm{tr}(M_x \varrho)$?

1) Collect data: Determine frequencies $\lambda_{\mathbf{x}}$ of the outcomes $\mathbf{x} = (x_1, \ldots, x_n)$. $(x_i \in \mathcal{X} \ \forall i)$

⟶ Law of large numbers: $\lambda_{\mathbf{x}} \xrightarrow{n \to \infty} p_X$ ⟵ iid

$\lambda_{\vec{x}}(y) = \frac{1}{n} \left| \{ i : x_i = y \} \right|$

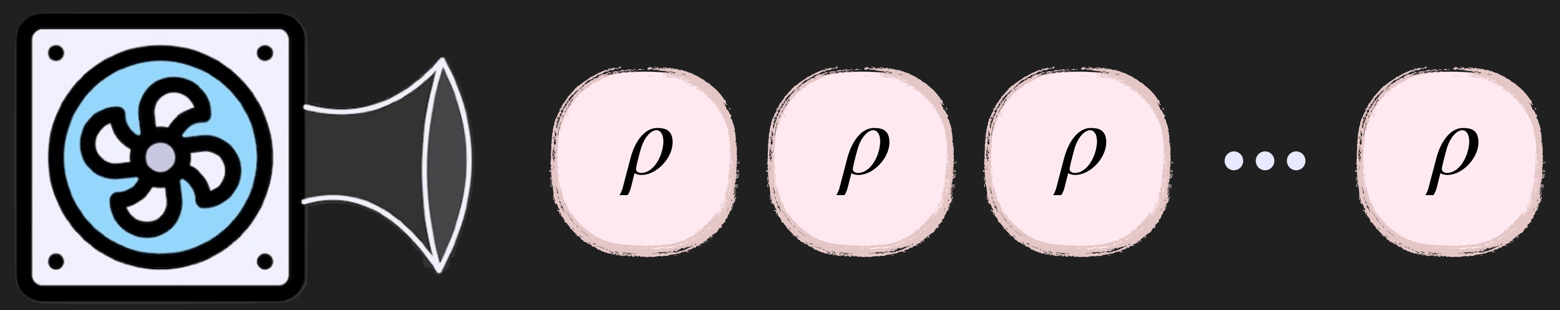
↳ distribution (for fixed $\vec{x}$)

2) Infer $P_X(x)$ from $\lambda_{\mathbf{x}}$.

BUT: There exist other states $\rho^{(n)} \neq \rho^{\otimes n}$ leading to the same $\lambda_{\mathbf{x}}$ (asymptotically)!

[1] Renner; Nature Physics (2007)

Motivation



$\underbrace{\rho \ \rho \ \rho \cdots \rho}^{\text{g}^{(n)}} \longrightarrow \rho^{(n)} \overset{\downarrow \text{ common assumption}}{\underset{?}{=}} \boxed{\rho^{\otimes n}}$ (iid)

Today!

BUT: There exist other states $\rho^{(n)} \neq \rho^{\otimes n}$ leading to the same λ_{x} (asymptotically)!

Why do we care?

- Information theory techniques $\longrightarrow$ often based on iid assumptions $\longrightarrow$ Robust ?
- We often work with the whole object $\rho^{(n)}$ $\longrightarrow$ example:
  - Entanglement measures defined via rates (E_C, E_D)
  - Regularized quantities
- Are non-iid resources as effective as iid resources?

What is $g^{(n)}$?

$\Rightarrow$ Example for $\rho^{(n)}$: Almost-iid states ! ($\leftarrow$ exponential de Finetti theorems [1])

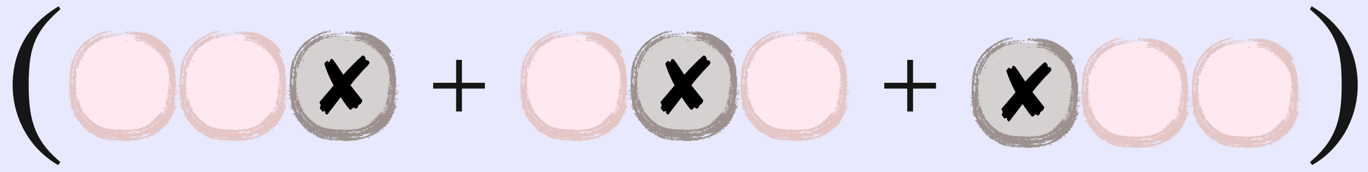
[2] Renner; PhD Diss. (2005)

Definition of almost-iid states

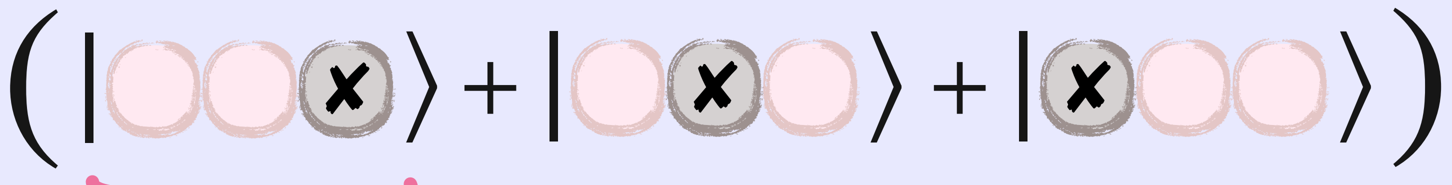
(arXiv: 2603.15792)

$$\rho^{\otimes n}$$

$$\sum_j p_j\, \pi_j\left[\rho^{\otimes n-r} \otimes \omega_j^{(r)}\right]\pi_j^\dagger$$

$\frac{1}{3}$ 

$$\sum_s \gamma_s\, \pi_s\left[\,|\theta\rangle^{\otimes n-r} \otimes |\omega_s^{(r)}\rangle\right]$$

$\frac{1}{\sqrt{3}}$ 

$$\mathcal{V}(\mathcal{H}^{\otimes n}, |\theta\rangle^{\otimes m}) := \{\pi(|\theta\rangle^{\otimes m} \otimes |\Omega^{(n-m)}\rangle) : \pi \in \mathcal{S}_n, |\Omega^{(n-m)}\rangle \in \mathcal{H}^{\otimes n-m}\}$$

Definition (special case [2]):

- $|\psi^{(n)}_{r,\theta}\rangle$ is an $\binom{n}{r}$-almost-iid state along $|\theta\rangle$ if:
  1) $|\psi^{(n)}_{r,\theta}\rangle$ is permutation invariant
  2) $|\psi^{(n)}_{r,\theta}\rangle \in \mathrm{span}\left[\mathcal{V}(\mathcal{H}^{\otimes n}, |\theta\rangle^{\otimes(n-r)})\right]$

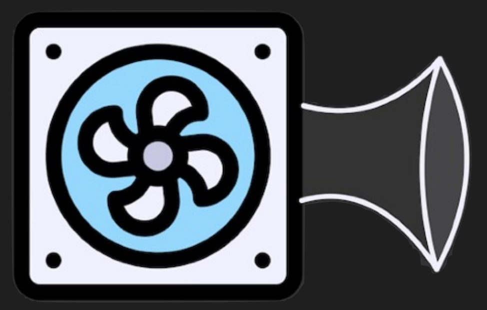
(statistics [2])

$$\lambda_x \xrightarrow{n\to\infty} p_X = \mathrm{tr}\left(M_{(.)}\rho\right)$$ ✓

⚠ almost iid states: $\rho^{(n)} \not\approx \rho^{\otimes n}$

(arXiv: 2603.15792)

Robustness of information measures



ρ ρ ρ $\cdots$ ρ $\longrightarrow$ $\rho^{(n)}$ an $\binom{n}{r}$-almost-iid state in ρ, $r = o(n)$

Entropy

$$H(\rho) = -\operatorname{tr}(\rho \log(\rho))$$

$$\frac{1}{n} H(\rho^{\otimes n}) \quad \text{vs.} \quad \frac{1}{n} H(\rho^{(n)})$$

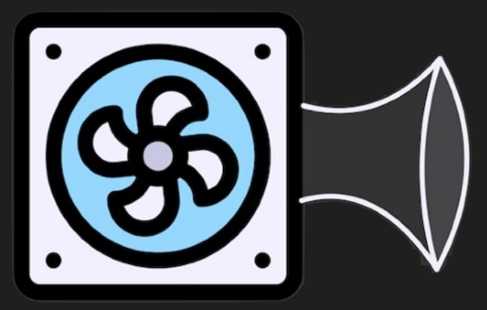
additive $\longrightarrow$ $\frac{1}{n} H(\rho^{\otimes n}) = H(\rho)$

$$\frac{1}{n} H(\rho^{(n)}) \xrightarrow{n \to \infty} \; ?$$

[3] Christandl-Winter; JMP (2004)

(arXiv: 2603.15792)

Robustness of information measures

 $\rho \quad \rho \quad \rho \cdots \rho \longrightarrow \rho^{(n)}$ an $\binom{n}{r}$-almost-iid state in ρ, $r = o(n)$

Entropy

$$H(\rho) = -\mathrm{tr}(\rho \log(\rho))$$

$$\frac{1}{n} H(\rho^{\otimes n}) \quad \text{vs.} \quad \frac{1}{n} H(\rho^{(n)})$$

$$\overset{\text{additive}}{=} H(\rho) \xleftarrow{n \to \infty} \frac{1}{n} H(\rho^{(n)})$$

$\Longrightarrow$

Squashed entanglement [3]

$$E_{sq}(\rho_{AB}) = \inf_{\rho_{ABE}} \frac{1}{2} I(A:B|E)_\rho$$

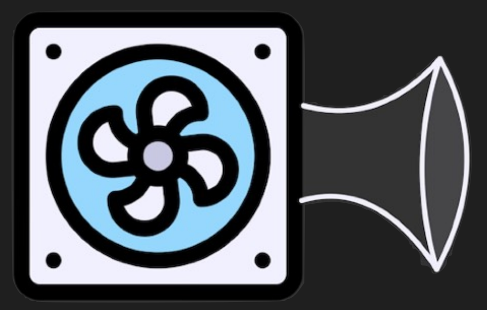
$$\frac{1}{n} E_{sq}(\rho_{AB}^{\otimes n}) \quad \text{vs.} \quad \frac{1}{n} E_{sq}(\rho_{AB}^{(n)})$$

$$\overset{\text{additive}}{=} E_{sq}(\rho_{AB}) \xleftarrow{n \to \infty} \frac{1}{n} E_{sq}(\rho_{AB}^{(n)})$$

$\Longrightarrow$ What about other measures, e.g., entanglement cost E_C and distillation E_D ?

Robustness of information measures

(arXiv: 2603.15792)

 $\rho\ \rho\ \rho \cdots \rho \longrightarrow \rho^{(n)}$ an $\binom{n}{r}$-almost-iid state in ρ, $r = o(n)$

$\Rightarrow$ What about other measures, e.g., entanglement cost E_C and distillation E_D ?

Unknown !

$\Rightarrow$ Another crucial point: Definition of a rate $\frac{m}{n}$: $\rho^{\otimes n} \longrightarrow \sigma^{\otimes m}$

$\longrightarrow$ based on an iid assumption ! (unlike squashed entanglement)

Goal Modify the definition of state transformation rates $\longrightarrow$ operational !

Operational state transformation rates

$$\rho^{(n)} \longrightarrow \sigma^{(m)}$$

State transformation rates

free operations

asymptotically close (in trace-distance)*

Traditional: $$R_{\mathscr{F}}(\rho \to \sigma) = \sup\left\{ r \;\middle|\; \exists\, \boldsymbol{\mathcal{P}}^{(r)} \in \mathscr{F} \;:\; \boldsymbol{\mathcal{P}}^{(r)}\left(\boldsymbol{\rho}^{\text{iid}}\right) \simeq \boldsymbol{\sigma}^{\text{iid}} \right\}$$

sequence $(\mathcal{P}_n^{(r)})_{n\in\mathbb{N}}$ s.t. $\mathcal{P}_n^{(r)}: \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes\lceil rn\rceil}$

(iid) sequence $(\rho^{\otimes n})_{n\in\mathbb{N}}$

$(\sigma^{\otimes\lceil rn\rceil})_{n\in\mathbb{N}}$

What if $\rho^{(n)} \approx_\epsilon \rho^{\otimes n}$? ok ✓

What if we have a defect: $\rho^{(n)} = \rho^{\otimes n-1} \otimes \omega$? ok ✓

What if we have a defect, but we don't know where ? not clear :/

What if we have superpositions of defect (almost-iid states) ? even worse :(

What if $\rho^{(n)}$ even more general ? even even worse !

*: error ε_n is sub-exponentially decreasing in n

State transformation rates – Definition

Operational rate (intuitive): $\rho^{(n)} = \rho\ \rho\ \rho \cdots \rho \xrightarrow{\text{tom}(\cdot)} \hat{\rho} \approx \rho$

Operational set (informal):

$$\mathrm{OP}(\mathtt{tom}(\cdot);\rho) := \left\{ \left(\rho^{(n)}\right)_{n\in\mathbb{N}} \,\middle|\, \Pr\left[\mathtt{tom}(\rho^{(n)}) \approx \rho\right] \xrightarrow{n\to\infty} 1 \right\}$$

Assumptions on tom(·):

1 works for iid input $\rho^{(n)} = \rho^{\otimes n}$ (e.g., tomography with POVM $\mathcal{M}^{\otimes n}$)

2 consistent under random sampling

$\Rightarrow \left(\mathrm{tr}_m\left[\mathrm{Sym}^{n+m}\left(\rho^{(n+m)}\right)\right]\right)_{n\in\mathbb{N}} \in \mathrm{OP}(\mathrm{tom}(\cdot);\rho)$

3 robust under "easy" defects ($r = o(n)$)

$\left(\pi\left(\rho^{\otimes n-r} \otimes \omega^{(r)}\right)\pi^{+}\right)_{n\in\mathbb{N}} \in \mathrm{OP}(\mathrm{tom}(\cdot);\rho)$

$\Rightarrow$ almost-iid states in ρ $\in \mathrm{OP}(\mathrm{tom}(\cdot);\rho)$ ✓

must include $\begin{cases} \cdot \text{ random permutations} \\ \cdot \text{ partial trace} \end{cases}$

Operational rate:

$$R^{\mathtt{tom}}_{\mathscr{F}}(\rho\to\sigma) = \sup\left\{ r \,\middle|\, \exists\, \boldsymbol{\mathcal{P}}^{(r)} \in \mathscr{F} \,:\, \boldsymbol{\mathcal{P}}^{(r)}\left(\overset{\text{sym}}{\mathrm{OP}}(\mathtt{tom}(\cdot);\rho)\right) \subseteq \overset{\text{sym}}{\mathrm{OP}}(\mathtt{tom}(\cdot);\sigma) \right\}$$

Main results

Theorem: (Characterising the operational set). *Let* $\texttt{tom}(\cdot)$ *be any sensible tomography protocol. Let* $\boldsymbol{\rho} \in \mathrm{OP}(\texttt{tom}(\cdot);\rho)$ *and assume* $\boldsymbol{\rho}$ *to be permutation invariant. Then, for any* $\alpha \in (0,1)$, *we find*

error sub-exp. in n

$\{(\varsigma_n)_n : \varsigma_n \text{ is } \binom{n}{r}\text{-almost-iid in } \varsigma,\ r = \lceil \lfloor n^{\kappa} \rceil \}$ $\kappa \in [0,1)$

$$(\mathrm{tr}_{\lceil n^{\alpha} \rceil})_{n \in \mathbb{N}} \leftarrow \mathbf{tr}_{\alpha}(\boldsymbol{\rho}) \simeq \boldsymbol{\tau} \in \mathcal{A}(\rho)\,.$$

Corollary: (Characterising the operational transformation rate). *For a given sensible tomography protocol* $\texttt{tom}(\cdot)$, *set of (free) operations* $\mathscr{F}$, *and any two density operators* ρ *and* σ, *we find*

↳ must include { · random permutations · partial trace

$$R^{\texttt{tom}}_{\mathscr{F}}(\rho \to \sigma) = \sup\left\{ r \mid \exists\, \boldsymbol{\mathcal{P}}^{(r)} \in \mathscr{F} : \boldsymbol{\mathcal{P}}^{(r)}(\mathcal{A}(\rho)) \underset{\sim}{\subseteq} \mathcal{A}(\sigma) \right\} =: \tilde{R}_{\mathscr{F}}(\rho \to \sigma)\,.$$

↳ error sub-exp. in n

Extra: $\tilde{R}_{\mathscr{F}}(\rho \to \sigma) = \sup\left\{ r \mid \exists\, \boldsymbol{\mathcal{P}}^{(r)} \in \mathscr{F} : \boldsymbol{\mathcal{P}}^{(r)}(\boldsymbol{\rho}^{\mathrm{iid}}) \underset{\sim}{\subseteq} \mathcal{A}(\sigma) \right\}$

$$\Rightarrow \boxed{R_{\mathscr{F}}(\varsigma \to \tau) \leq \tilde{R}_{\mathscr{F}}(\varsigma \to \tau)}$$

Main results – special cases

Extra: $\tilde{R}_{\mathscr{F}}(\rho \to \sigma) = \sup\left\{ r \,\middle|\, \exists\, \boldsymbol{\mathcal{P}}^{(r)} \in \mathscr{F} \,:\, \boldsymbol{\mathcal{P}}^{(r)}(\boldsymbol{\rho}^{\mathrm{iid}}) \underset{\sim}{\subseteq} \mathcal{A}(\sigma) \right\}$

$\Rightarrow \boxed{R_{\mathscr{F}}(\rho \to \sigma) \leq \tilde{R}_{\mathscr{F}}(\rho \to \sigma)}$ (Δ)

Entanglement distillation: $\tilde{E}_D(\rho) = \tilde{R}\big(\rho \to |\Phi^+\rangle\langle\Phi^+|\big)$

$|\Phi^+\rangle$: Bell state

$(\Delta) \Rightarrow E_D(\rho) \leq \tilde{E}_D(\rho)$ ✓

$(\square)$ $E_D(\rho) \geq \tilde{E}_D(\rho)$ ✓

$\Rightarrow E_D(\rho) = \tilde{E}_D(\rho)$

Entanglement cost: $\tilde{E}_C(\rho) = \dfrac{1}{\tilde{R}(|\Phi^+\rangle\langle\Phi^+| \to \rho)}$

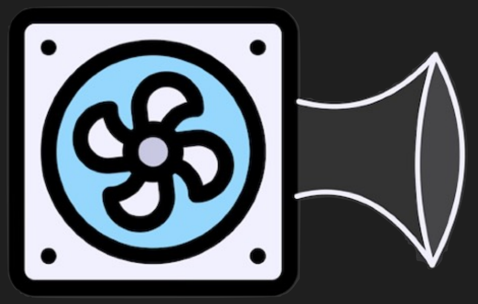
$(\Delta) \Rightarrow \tilde{E}_C(\rho) \leq E_C(\rho)$ ✓

but $\tilde{E}_C(\rho) \overset{?}{\geq} E_C(\rho)$

if true: :-)

if wrong: :-O

Conclusion

TODAY:  $\rho\ \rho\ \rho \cdots \rho \longrightarrow \rho^{(n)} \not\approx \rho^{\otimes n}$

de Finetti-type theorem

$$\mathbf{tr}_\alpha(\boldsymbol{\rho}) \simeq \boldsymbol{\tau} \in \mathcal{A}(\rho)$$

$\downarrow$

Notion of almost-iid states

$$\frac{1}{\sqrt{3}}\Big(|\circ\circ\otimes\rangle + |\circ\otimes\circ\rangle + |\otimes\circ\circ\rangle\Big)$$

(arXiv: 2603.15792)

Operational state transformation rates

Traditional: $\rho^{\otimes n} \longrightarrow \sigma^{\otimes m}$

Operational: $\rho^{(n)} \longrightarrow \sigma^{(m)}$

$\Updownarrow$ $\quad \rho^{(n)} \xrightarrow{\text{tom}(\cdot)} \rho \qquad \sigma^{(m)} \xrightarrow{\text{tom}(\cdot)} \sigma$

Main result: $\rho^{\otimes n} \longrightarrow \sigma^{(m)}$ (almost-iid in σ)

$\Rightarrow \boxed{R_{\mathcal{F}}(\rho\to\sigma) \leq \tilde{R}_{\mathcal{F}}(\rho\to\sigma)}$

but $R_{\mathcal{F}}(\rho\to\sigma) \overset{?}{=} \tilde{R}_{\mathcal{F}}(\rho\to\sigma)$

Thank you for your attention!