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Thermal Operations from Informational Equilibrium

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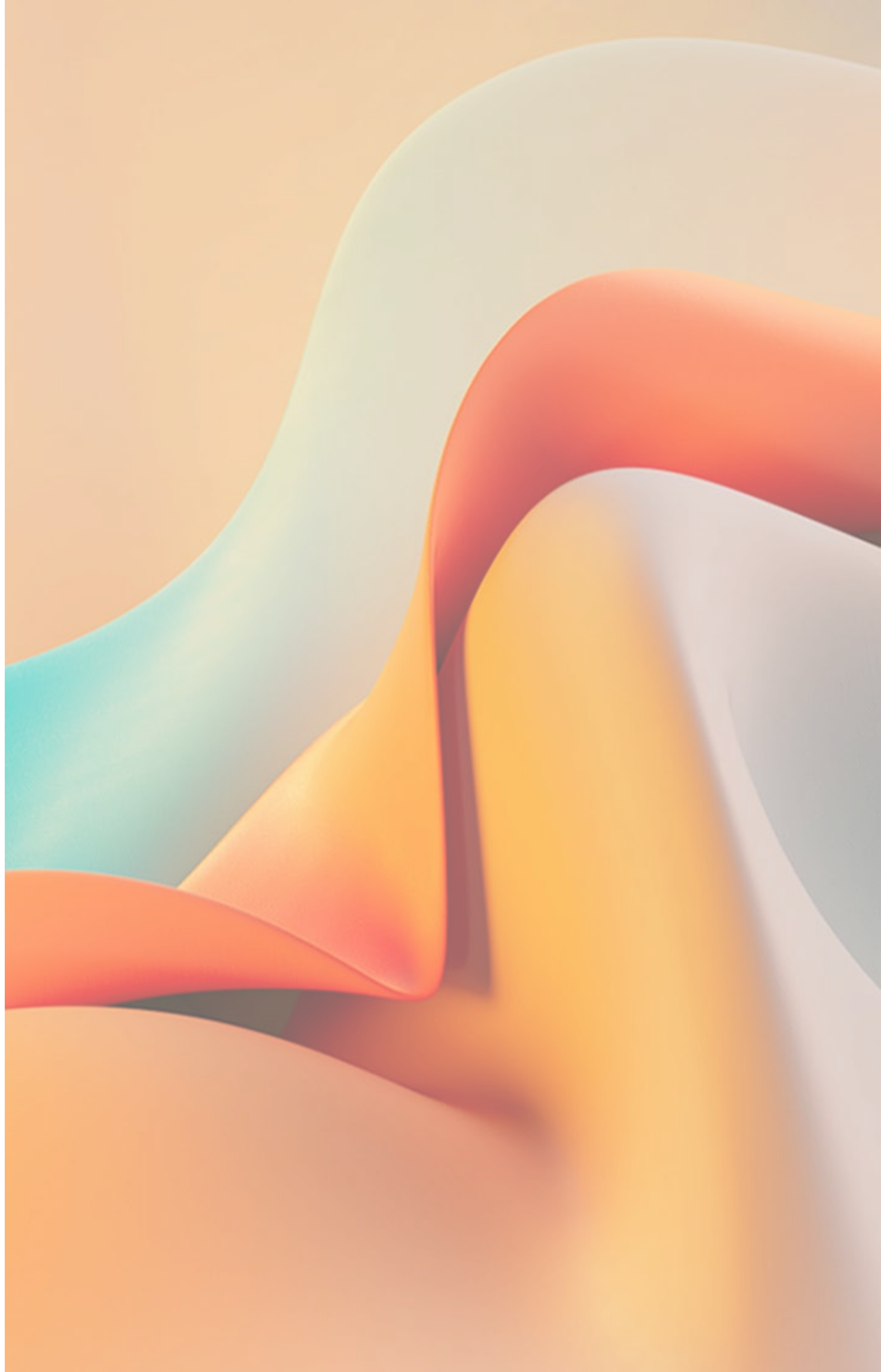
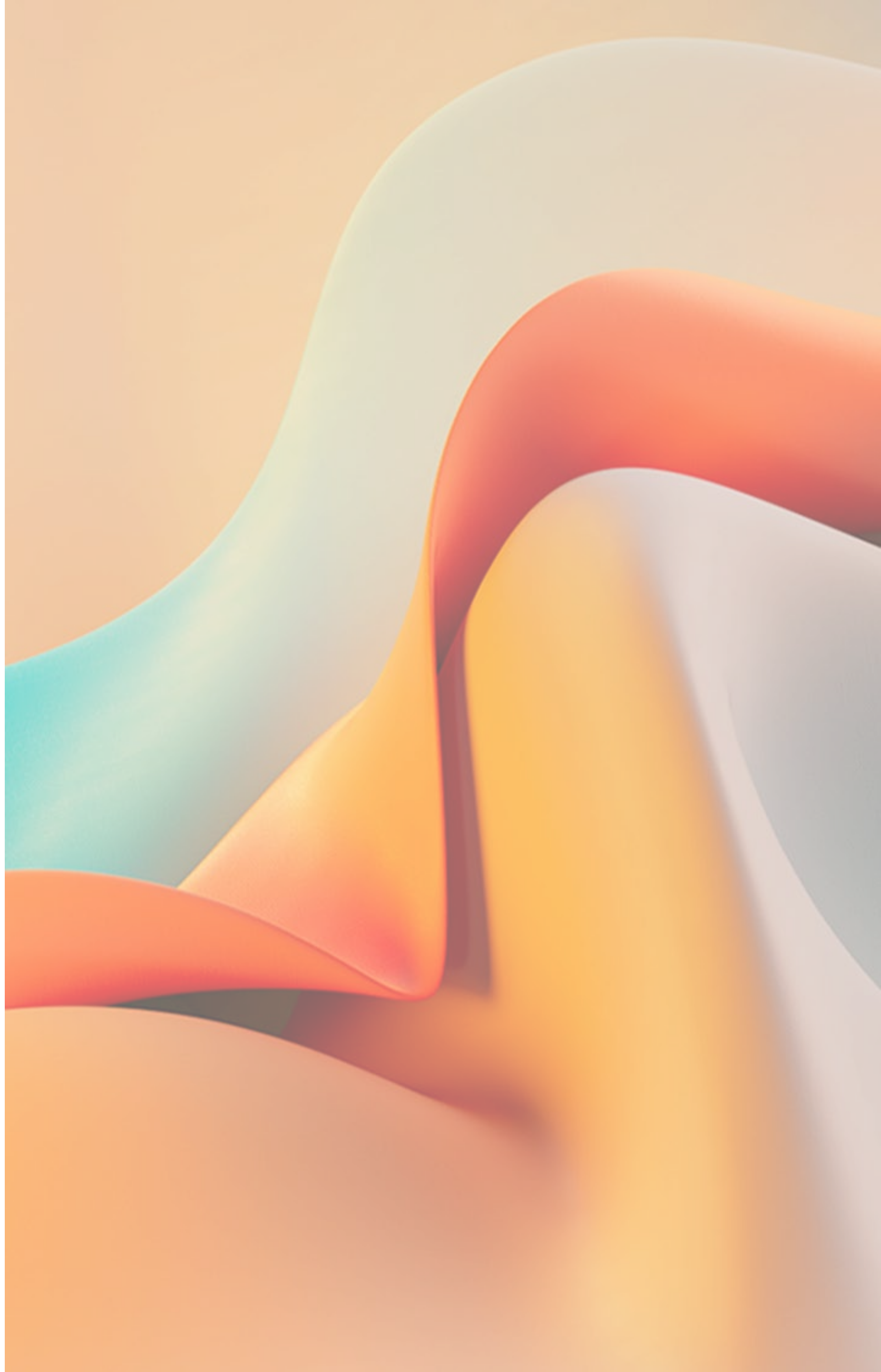


Understanding thermal phenomena seems to require thorough understanding of the **energetics** of quantum systems;

Can we understand the **fundamentals of quantum thermodynamics** through the lens of **information theory**?



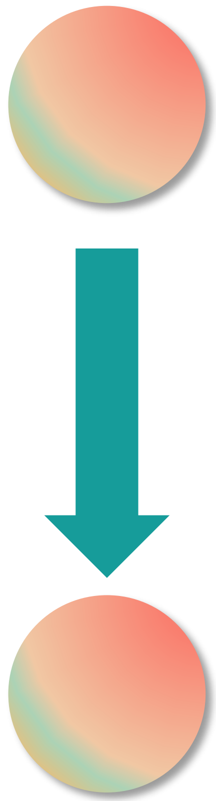
Thermal Operations vs. Gibbs-Preserving maps

Thermodynamic equilibrium

- In quantum thermodynamics, a system that reached the thermal equilibrium is assumed to be in the **Gibbs state**:

$$\gamma = \frac{1}{Z} e^{-\beta H}$$

- $Z = \mathrm{Tr}[e^{-\beta H}]$ and $\beta = 1/kT$ is the inverse temperature
- A physical process that "equilibrates" a system's state is called an **equilibration**
- But there are **many different** notions of equilibration in quantum thermodynamics



Gibbs-Preserving (GP) maps vs Thermal Operations (TO)

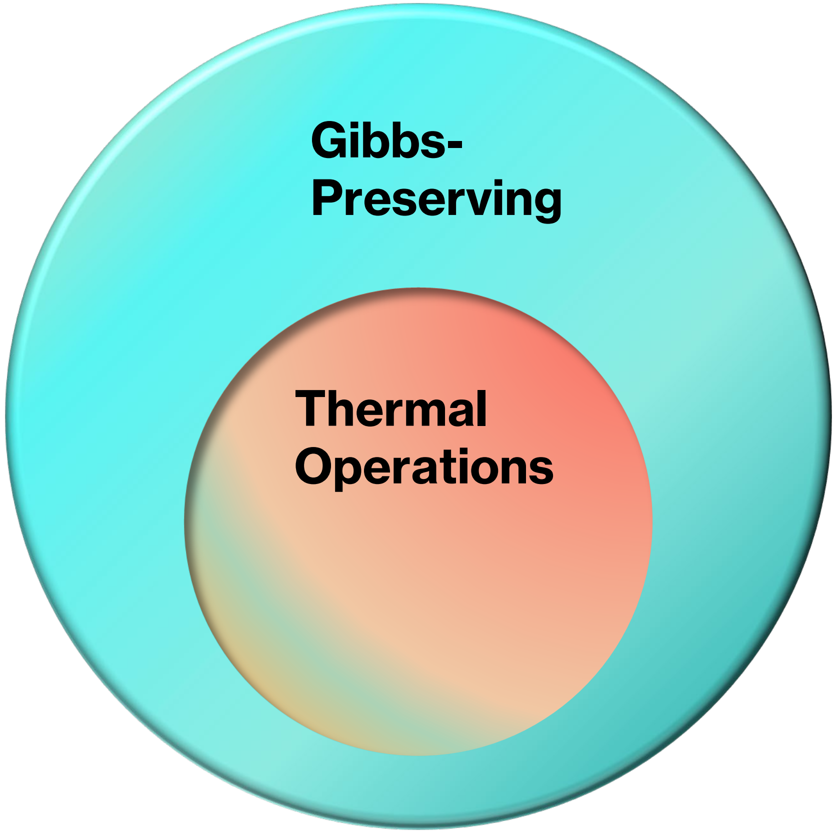
- **Gibbs-Preserving map** Φ whose sole condition is preserving Gibbs states:

$$\Phi(\gamma) = \gamma$$

- **Thermal Operation** (TO) Ψ such that

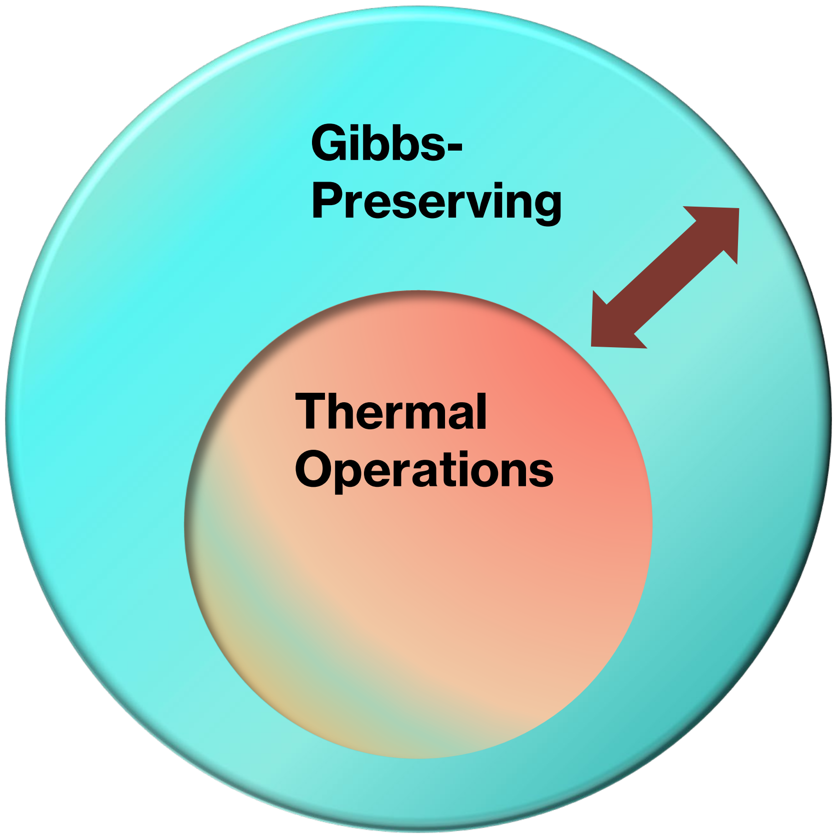
$$\Psi(\rho) = \mathrm{Tr}_E\left[U(\rho_A \otimes \gamma_E)U^\dagger\right]$$

where E is environment with an energy preserving unitary U on AE.



Gibbs-Preserving (GP) maps vs Thermal Operations (TO)

- **GP is mathematically easier** to deal with, but lacks a clear physical interpretation
- **TO is operationally more meaningful**, but lacks nice mathematical properties
- $TO \subseteq GP$
- It has been well-known that the inclusion relation is proper $(TO \subsetneq GP)$
- Most famously, GP maps can **generate quantum coherence** whereas TOs cannot!



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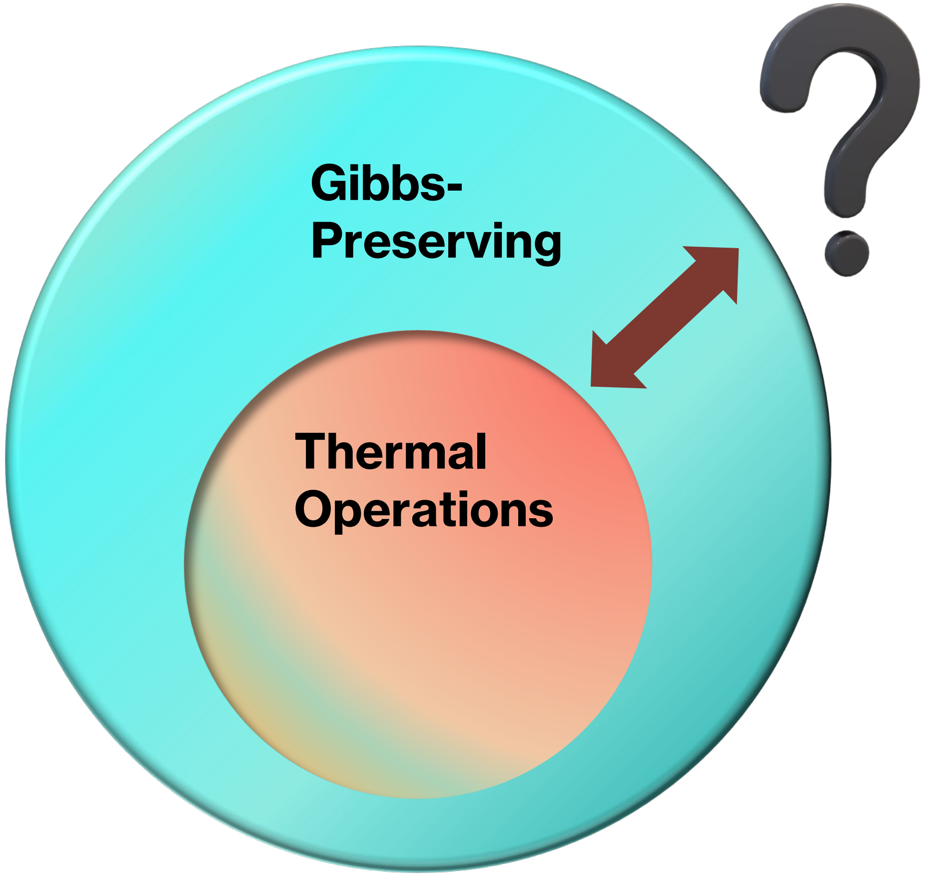
Gibbs-Preserving (GP) maps vs Thermal Operations (TO)

- However, it is still **not very clear which physical principle** independent of the choice of specific Hamiltonian and temperature **separates them!**



Question:

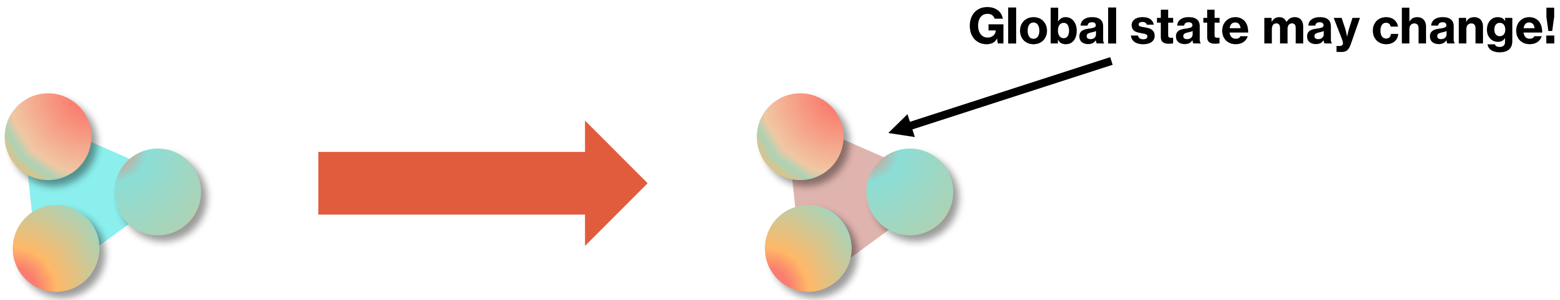
Which principle independent of energetics can explain this separation?

Informational Characterization of Thermal Operations

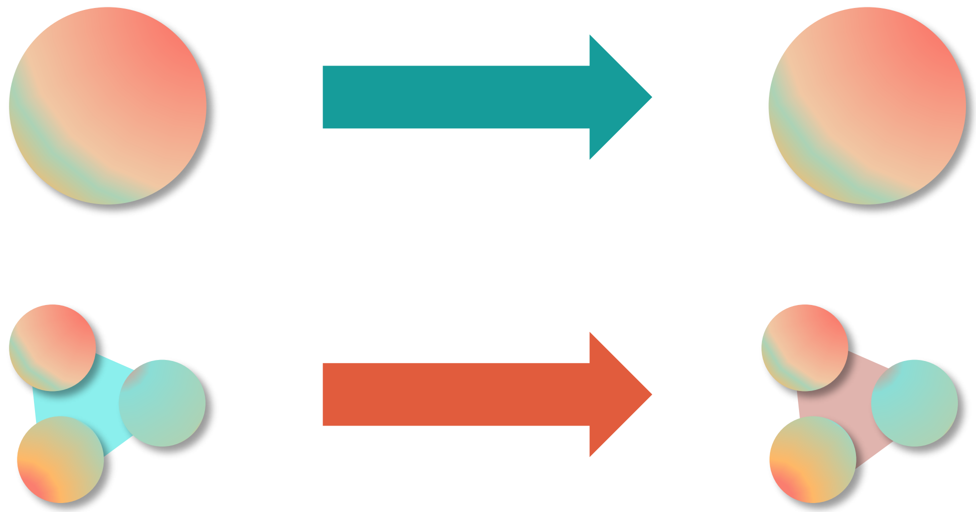
Different notions of equilibrium

- **Equilibrium** is a central notion in physics that has a few manifestations:
- (*Stationarity*) First, a **single system** is said to be at equilibrium if its state remains invariant under its autonomous time evolution.

- (*Relation*) Second, **multiple systems** are said to be at equilibrium if their local states remain invariant under their **interaction.**



Equilibration



- A process that preserves the equilibrium state (thus drive non-equilibrium states to equilibrium) is called an **equilibration.**

Information theoretic characterization of equilibrium

- (local) **stationarity** vs (relational) **equilibrium**

- We say that ω is **stationary** with respect to U if $U\omega U^\dagger = \omega$
- On the other hand, we say that systems A and B in local states ω_A and ω_B are in **equilibrium** relative to U if

$$\mathrm{Tr}_B \sigma_{AB} = \omega_A \text{ and } \mathrm{Tr}_A \sigma_{AB} = \omega_B$$

where $\sigma_{AB} = U(\omega_A \otimes \omega_B)U^\dagger$.

How can we define informational equilibration?

Definition: Equilibrating dilation

Consider a quantum channel T on system A with a fixed point ω_A. We say that has an **equilibrating dilation** with respect to ω_A if there exists a dilation (U, ω_B) of T such that

$$T(\rho) = \mathrm{Tr}_B\left[U(\rho_A \otimes \omega_B)U^\dagger\right]$$

and

$$\mathrm{Tr}_A\left[U(\omega_A \otimes \omega_B)U^\dagger\right] = \omega_B.$$

A channel with an equilibrating dilation is called an **equilibration** or an **equilibrating channel**.

- It is a **fixed-point dilation**. If the system already **reached a fixed point**, then the ancillary system is **also required to have reached the fixed point**.

Informational Zeroth Law

Informational Zeroth Law

Suppose $\omega_A$and ω_B are in equilibrium relative to U. Then $U(\omega_A \otimes \omega_B)U^\dagger = \omega_A \otimes \omega_B$.

- This result **unifies the two notions of equilibrium**; the relational equilibrium between A and B implies stationarity of the composite system AB.
- It **can be generalized to multipartite** systems $(\omega_A \otimes \omega_B \otimes \omega_C \otimes \cdots)$

Thermal Operation from informational equilibrium

Main Result 1

Let a channel T on system A admit an equilibrating dilation (U_{AB}, ω_B) with respect to ω_A.
Then:

1. $[U, \omega_A \otimes \omega_B] = 0$,
2. For every $t \in \mathbb{R}$ and every state ρ_A on A we have $T\left(\omega_A^{it} \rho_A \omega_A^{-it}\right) = \omega_A^{it} T(\rho_A) \omega_A^{-it}$,
3. If ω_A has full rank, ω_B can be chosen to have full rank.

- Because of this result, one can interpret local fixed points ω_A and ω_B as Gibbs states for some **Hamiltonians** ($\omega_{A/B} = e^{\beta H_{A/B}} / Z_{A/B}$) for an appropriate $H_{A/B}$ and $Z_{A/B}$), so that U_{AB} is an **energy-preserving unitary**

Thermal Operation from informational equilibrium

- Therefore, any equilibrating channel T admits a dilation of the form

$$T(\rho) = \mathrm{Tr}_B\left[U(\rho_A \otimes \omega_B)U^\dagger\right]$$

with an energy preserving unitary U and a Gibbs state ω_B for **some** Hamiltonian. **Hence, it is a thermal operation.**

- The **converse is also true**; every TO is also equilibrating.

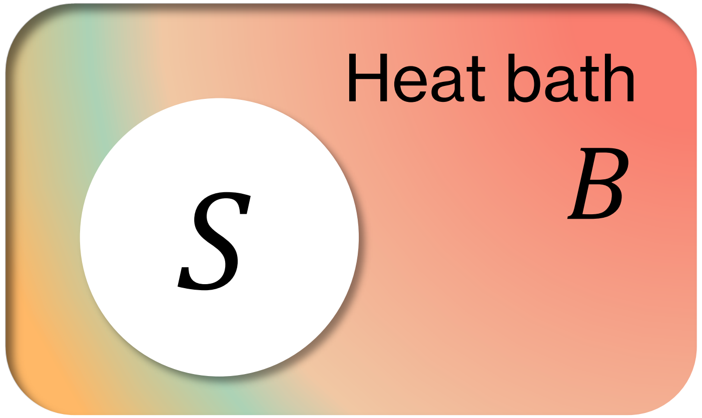
Thermal Operation from informational equilibrium

- This observation also **characterizes non-TO GP maps**
- They do not admit an equilibrating dilation, and as a result, whenever they are implemented with an interaction with environment, the state of the environment **must change** after the interaction, **even after reaching local equilibrium.**
- *Summary*: non-TO GP maps **actively generates non-equilibrium resources** in environment to maintain local equilibrium

Information Theoretic Idealization and Catalytic Channels

Informational idealization of heat bath

- A **heat bath** is a system which is assumed to have reached equilibrium and be large or thermalized fast enough so that it **stays in the same state after interaction**.
- What is **the information theoretic limit** of its behavior?



- We idealize it to a **catalyst**; a system who **remains the same after interaction**, independent of the initial state of the system in contact.

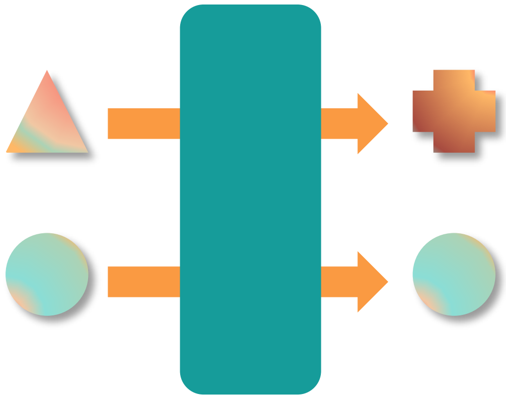
Catalysis in quantum information theory

Catalysis of Quantum Resources

Catalysis involve transformations using ancillary catalysts that remain unchanged after their operation.

Role of Catalysts

Catalysts enable transformations that would otherwise be impossible without their presence.



Catalytic channels

Definition: Catalytic dilation

A quantum channel T on system A has a **catalytic dilation** (U, ω_B) if

$$T(\rho) = \mathrm{Tr}_B\left[U(\rho_A \otimes \omega_B)U^\dagger\right]$$

and

$$\mathrm{Tr}_A\left[U(\rho_A \otimes \omega_B)U^\dagger\right] = \omega_B$$

for all states ρ_A on A. A channel with a catalytic dilation is called a **catalytic channel**.

- Difference from an equilibrating dilation : system B is required to stay in the same state ω_B **for any state** ρ_A of A, not just the fixed-point ω_A

Catalytic channels

- A catalyst **mimics the extreme limit of the behavior a large heat bath** by remaining absolutely intact after interaction.
- This characterization forces a catalyst to behave like a $T = \infty$ heat bath.
- So, it can only increase the entropy of systems in contact.
- **Every catalytic channel is a unital channel** $(T(\mathbb{I}) = \mathbb{I})$

Catalytic unitary

Fact

For any bipartite unitary operator U, (U,τ) can be a catalytic dilation for some catalyst state τ if and only if its **partial transpose** U^{T_B} **is also unitary.**

Such a unitary is called a **catalytic unitary.**

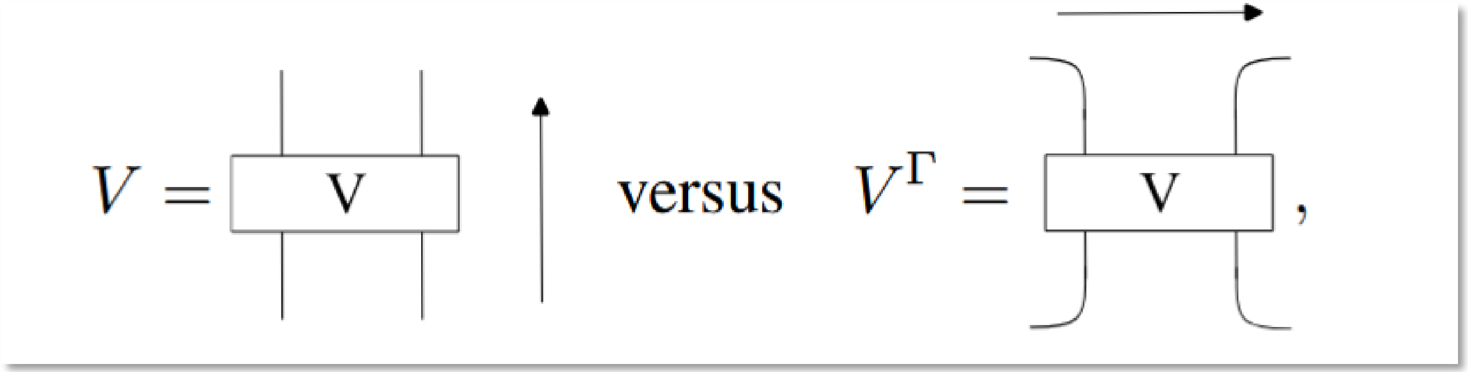
- This characterization serves as an easy-to-test criterion for catalytic unitaries.
- Maybe it has something to do with temporal compatibility of two systems?...

Catalytic unitary

Observation

A bipartite unitary U is catalytic if and only if $U\mathbb{S}$ is **dual-unitary** where $\mathbb{S}$ is the SWAP gate.

- A bipartite unitary U is called dual-unitary if its **reshuffled** version is also unitary:



- This concept is important in many-body physics, quantum chaos and black hole science. Catalytic unitary is also known as PT-unitary in that community.

Hierarchy of Unital Quantum Channels

Hierarchy of doubly-stochastic maps

- **Doubly-stochastic maps**(a.k.a. **unital maps**), channels that preserve the maximally mixed states, correspond to GP maps **for fully-degenerate Hamiltonians** or **infinite temperature**
- In classical statistical mechanics, every **doubly-stochastic map** is decomposed into **a convex combination of permutation maps** by the Birkhoff-von Neumann theorem.
- This is **not true** in quantum thermodynamics! It has a very **rich structure**!
- Studying the structure will reveal the nature of quantum mechanics.

ESCAPE BUTTON HERE

Mixed unitary

- Catalytic channels provide a fresh perspective on the **hierarchical structure** of unital maps
- First, the class of catalytic channels has an interesting **subclass**:

Definition: Mixed unitary (=Random unitary)

A quantum channel T is called a **mixed unitary channel** if there exist unitary operators U_i such that

$$T(\rho) = \sum_i p_i U_i \rho U_i^\dagger$$

with some probability distribution $\{p_i\}$, for all states ρ.

Mixed unitary

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with some probability distribution $\{p_i\}$, for all states ρ.

- This class is the **quantum counterpart** of **mixed permutations**.
- There are **non-mixed unitary doubly-stochastic** channels!

→ **Failure of the Birkhoff-von Neumann theorem in quantum mechanics**

Schur multiplier

Definition: Schur multiplier

A quantum channel T is called a **Schur multiplier channel** if there exists an $n \times n$ positive semidefinite matrix X with $X_{ii} = 1$ for $i = 1, 2, \dots, n$ such that

$$T(\rho) = \rho \circ X$$

where $\circ$ here denotes the Schur product $(\rho \circ X)_{ij} = \rho_{ij} X_{ij}$.

- Every Schur multiplier channel is unital.
- It models generalized **dephasing channels**.

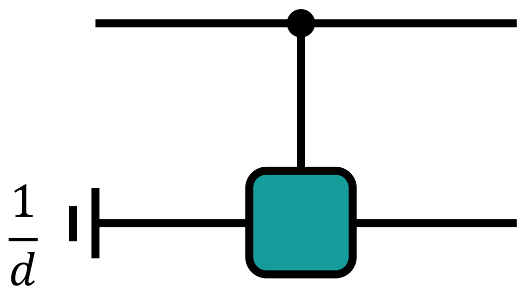
Bloch sphere

Real Schur multipliers are catalytic

Main Result 2

Every Schur multiplier channel $T(\rho) = \rho \circ X$ with a **real matrix** X (a matrix with real components) is **catalytic**

- Used the mathematical results found in [Haagerup and Musat (2011)] .
- All such Schur multiplier channels can be implemented with a **controlled unitary** with the **system of interest as the control**.



Not every catalytic channel is mixed unitary

Haagerup and Musat, Comm. In Math. Phys., (2011)

Example 3.3. Let $\beta = 1/\sqrt{5}$ and set

$$B: = \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & -\beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}.$$

We claim that T_B is a factorizable τ_6-Markov map on $M_6(\mathbb{C})$, but $T_B \notin \mathrm{conv}(\mathrm{Aut}(M_6(\mathbb{C})))$.

- B is a real matrix with unity diagonal components, so $\boldsymbol{T_B}$ **is catalytic**.
- But **it is not a mixed unitary channel**.
- So there exists a **non-mixed unitary but catalytic channel**.

Factorizable maps

Definition: Factorizable maps

A quantum channel T is called **factorizable** if it can be implemented with an ancillary system (possibly infinite dimensional) prepared in the maximally mixed state.

A quantum channel T is called **exactly factorizable** if the ancillary system is **finite dimensional**, i.e.

$$T(\rho) = \mathrm{Tr}_B U\left(\rho \otimes \frac{\mathbb{I}_B}{d_B}\right)U^\dagger$$

A quantum channel T is called **strongly factorizable** it is a convex combination of exactly factorizable maps.

- A factorizable map = a channel implementable with a **completely unknown ancilla**

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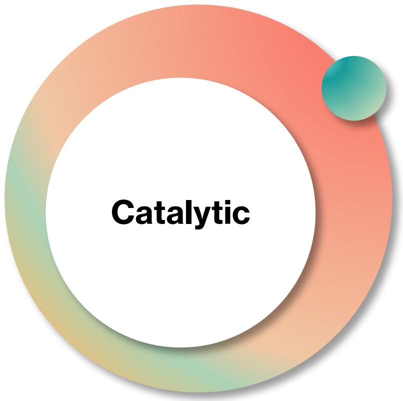
- By Carathéodory's theorem, it follows that a channel is **strongly factorizable** iff it is **equilibrating** and **unital**.

Non-catalytic factorizable maps

Main Result 3

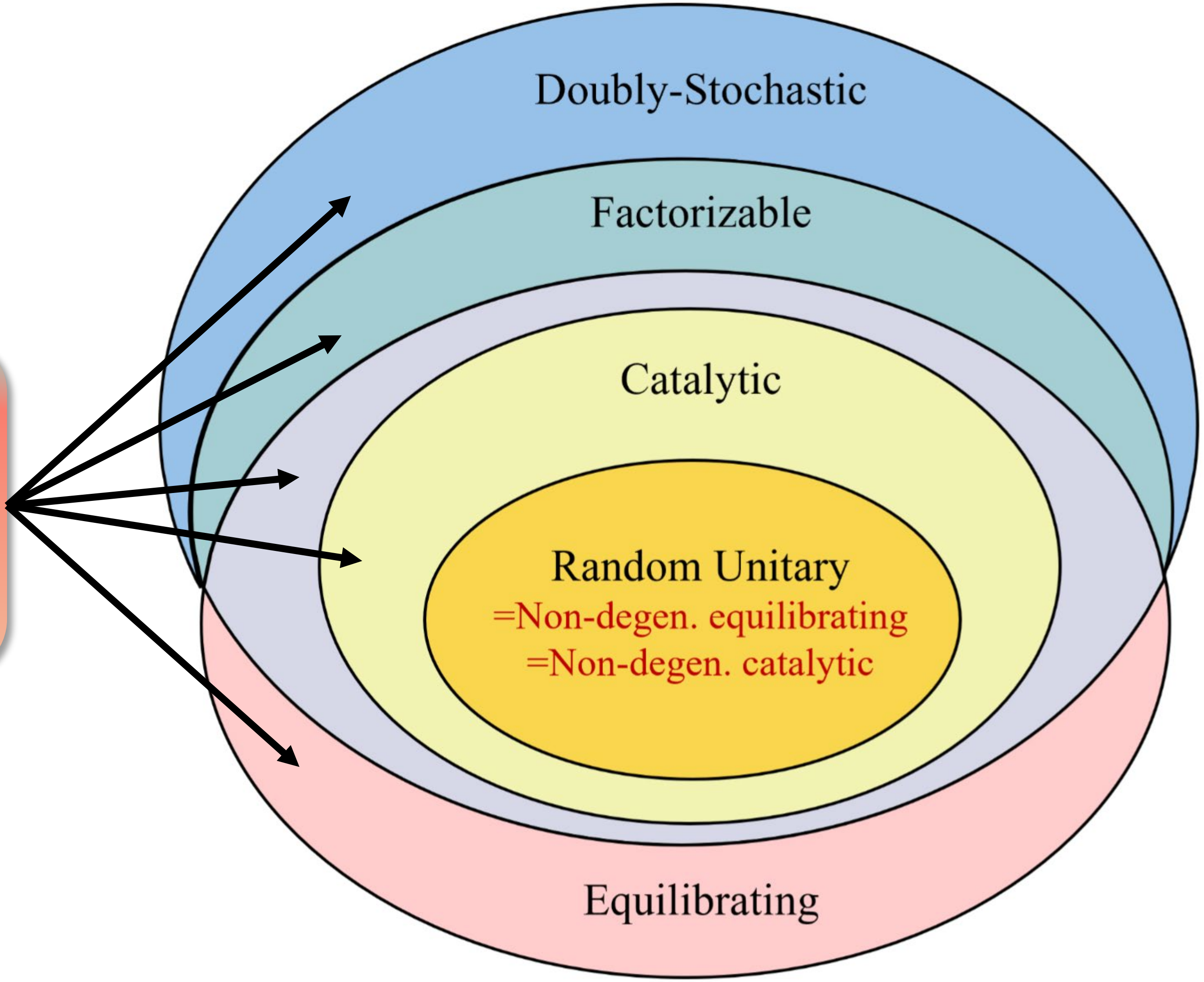
It is known that if a catalytic channel is **extremal** (cannot be expressed as a convex sum) in the set of unital channels, then it is unitary. However, there exists a non-unitary equilibrating channel that is **extremal** in the set of unital channels.



- This result shows that **not every equilibrating unital channel is catalytic**.
- **Sometimes in equilibration, non-equilibrium property cannot be destroyed but only can be relocated!!**

Hierarchy of doubly-stochastic maps

All inclusions here are proper!



Conclusion

Informational Equilibrium

Reconsidering equilibrium from the purely informational perspective, now we have **a new characterization of TO maps** independent of the notion of energy as ***an (informational) equilibrating process.***

Informational (extremal) Bath

Idealizing large **heat baths as an informational catalyst** introduces a new class of unital channels called *catalytic channels*. They form a new subclass that gives a **richer hierarchy of Gibbs-preserving maps** for fully degenerate Hamiltonians.

Open problems

Nonequilibrium measures

Develop measures of non-equilibrium costs of general GP maps based on the characterization given here

Hierarchy of general GP maps

Generalize the structure results to non-trivial Hamiltonians: What is the catalytic channels for them?

Characterization of catalytic channels

What would be the mathematical characterization of catalytic channels? Would it have a decomposition result?

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- On 'Temporal correlation operator-based enhancement of quantum metrology' and 'Hybrid quantum resource theories' @UNIST

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