A doubly composite Chernoff–Stein lemma and its applications

2510.06342 & 2510.06340

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Ludovico Lami

Scuola Normale Superiore, Pisa, Italy

European Research Council
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Introduction

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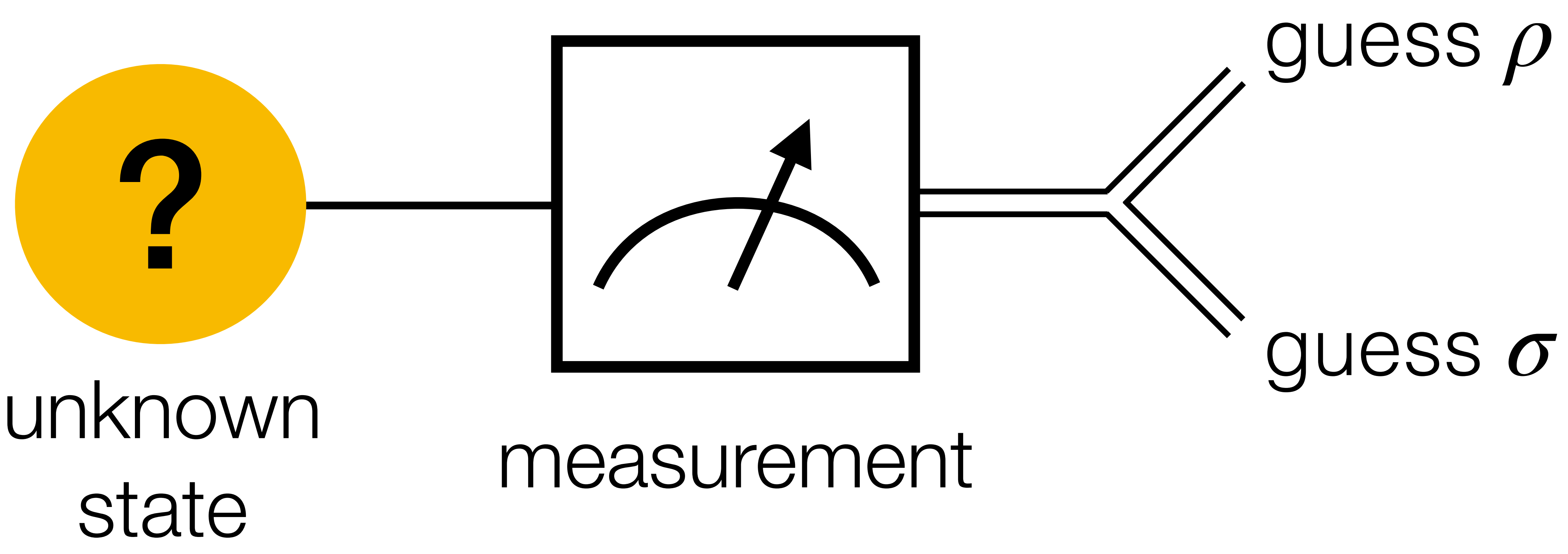
Alternative hypothesis: σ



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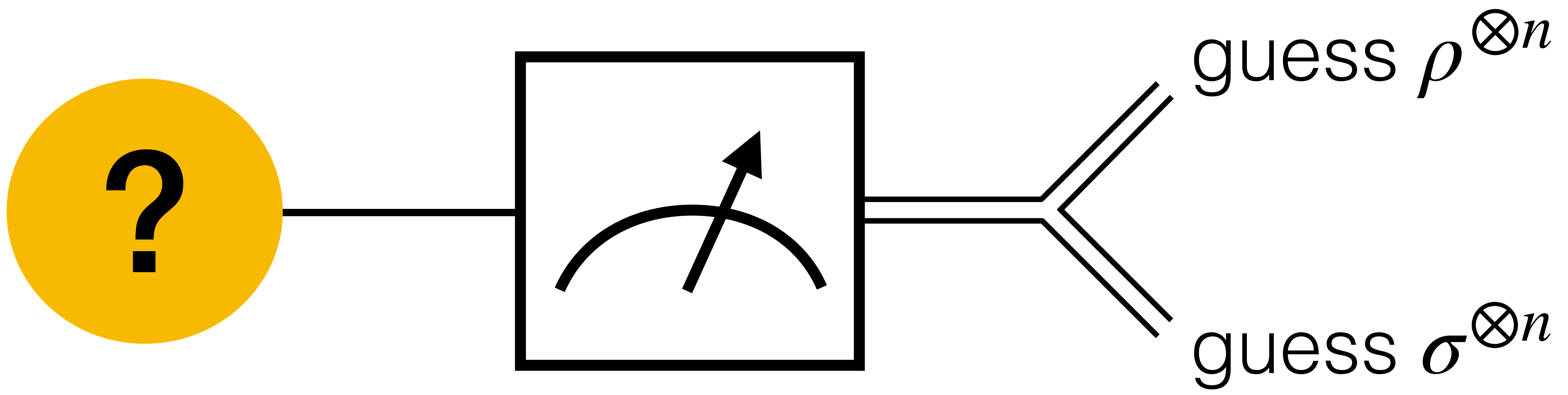
Simple IID hypotheses: many copies of one of two states.

Null hypothesis: $\rho^{\otimes n}$

Alternative hypothesis: $\sigma^{\otimes n}$



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Composite hypotheses: two **sets** of quantum states.

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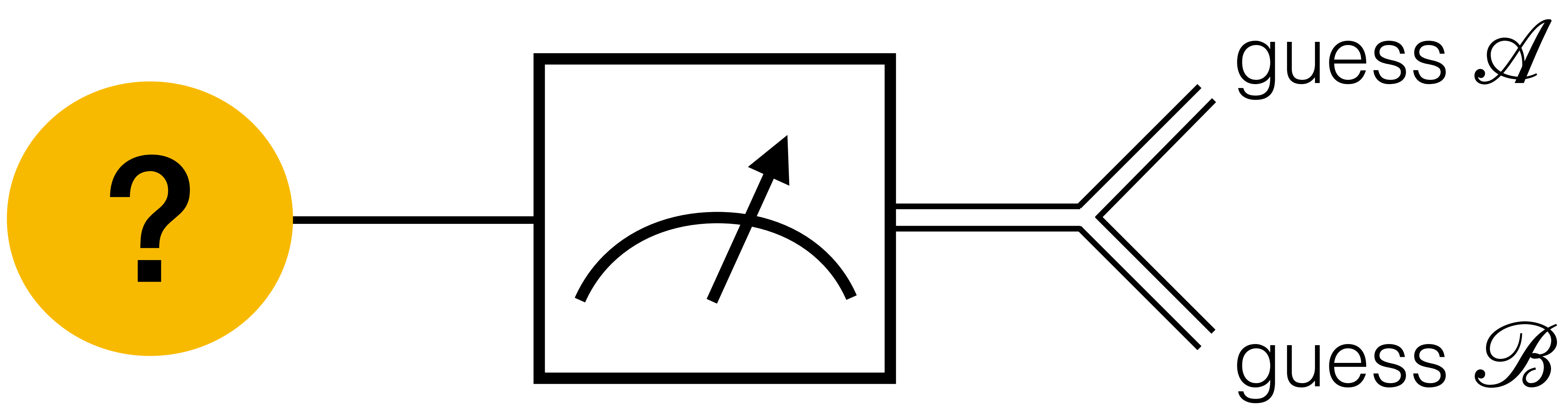

$\mathscr{A}, \mathscr{B}$: sets of states on the same quantum system.

Quantum hypothesis testing = Given two hypotheses concerning a quantum state, which one holds true?

Composite hypotheses: two sequences of sets of states, indexed by $n \in \mathbb{N}$.

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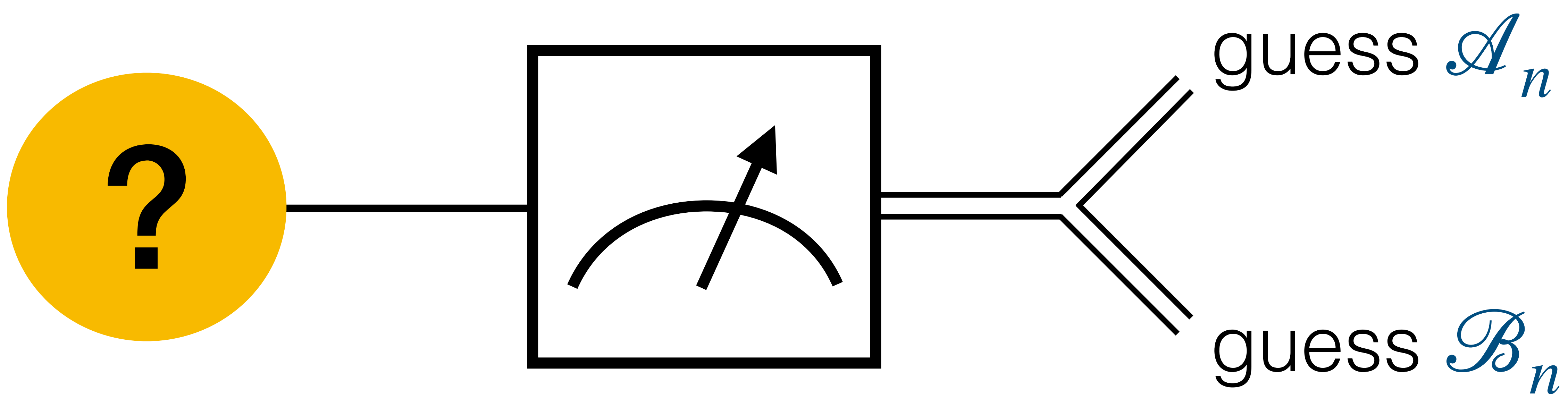

$\mathscr{A}_n, \mathscr{B}_n \subseteq \mathscr{D}\left(\mathscr{H}^{\otimes n}\right)$ sequences of sets of states on the n-copy system.

Two types of error:

- Type 1: null hypothesis correct, guessed alternative
- Type 2: alternative hypothesis correct, guessed null

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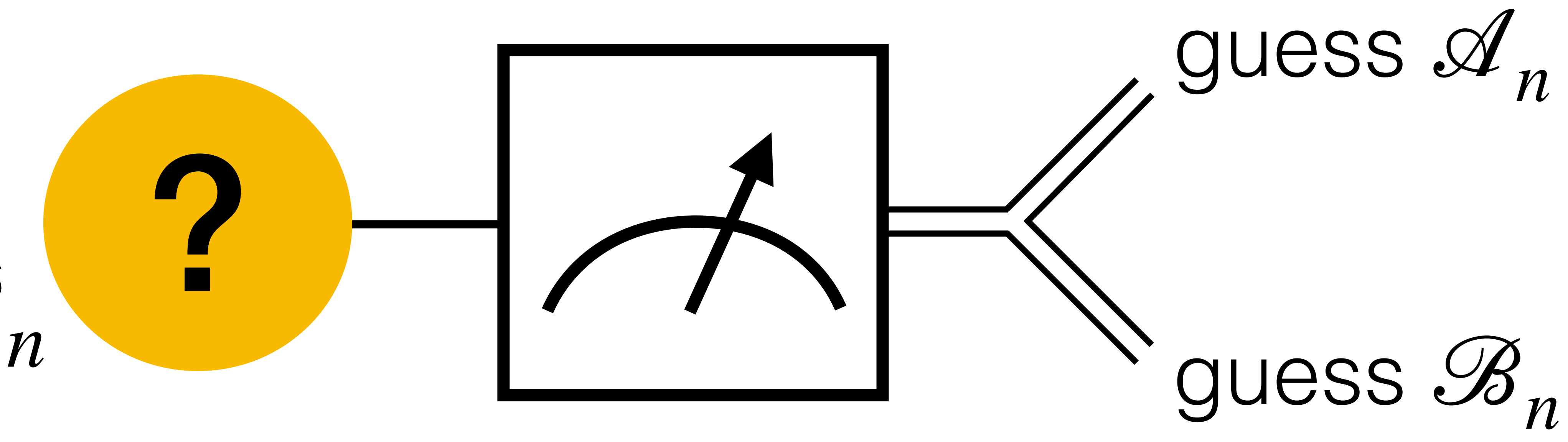

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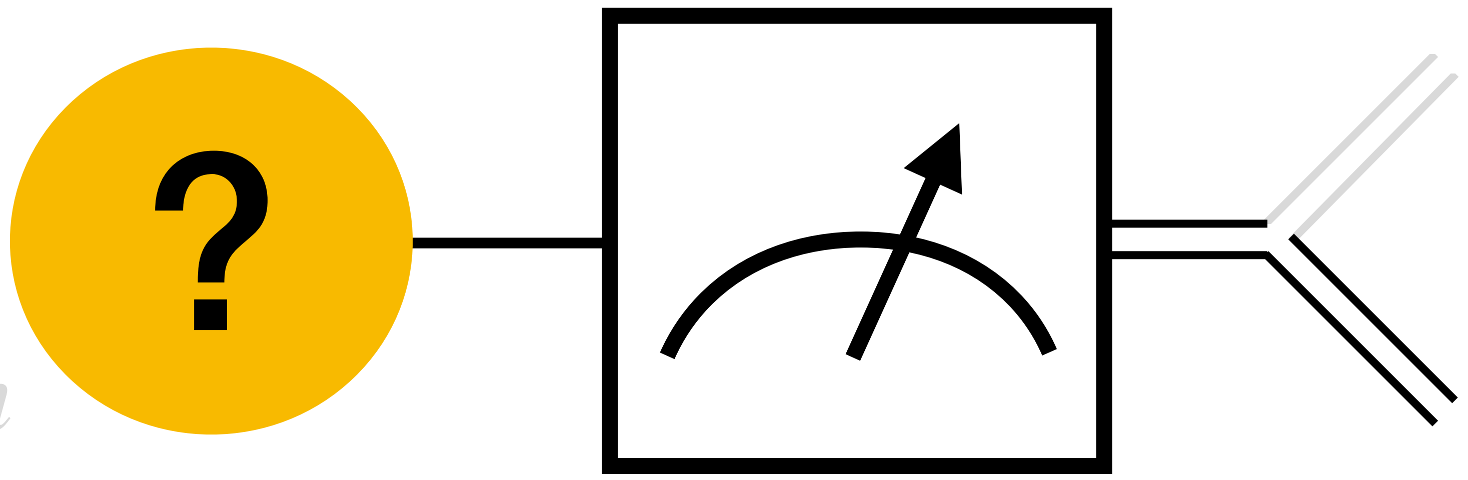

guess $\mathscr{A}_n$ ⇝ type 2 error

guess $\mathscr{B}_n$ ⇝ type 1 error

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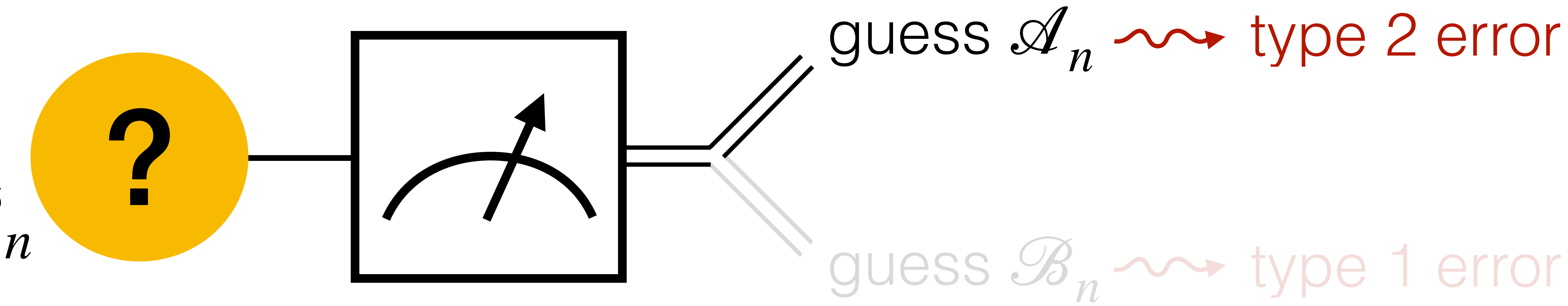

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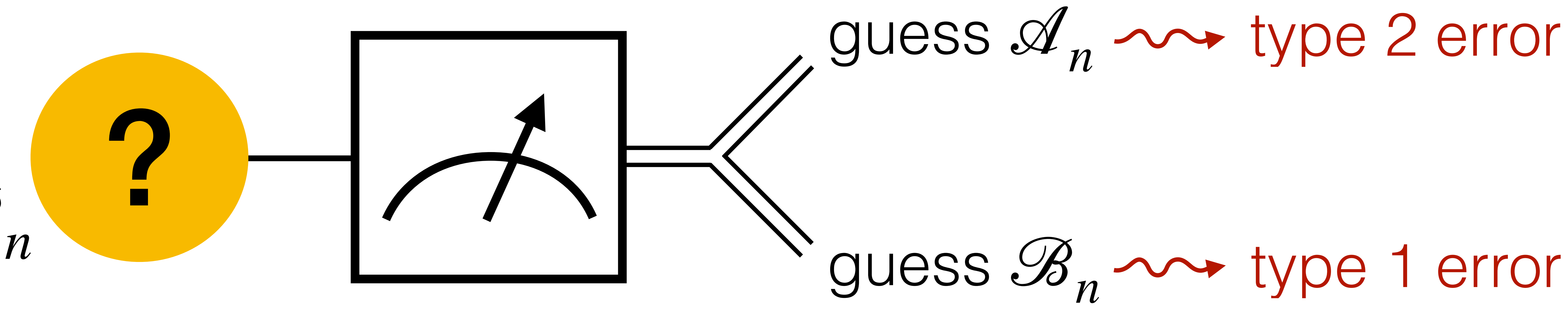

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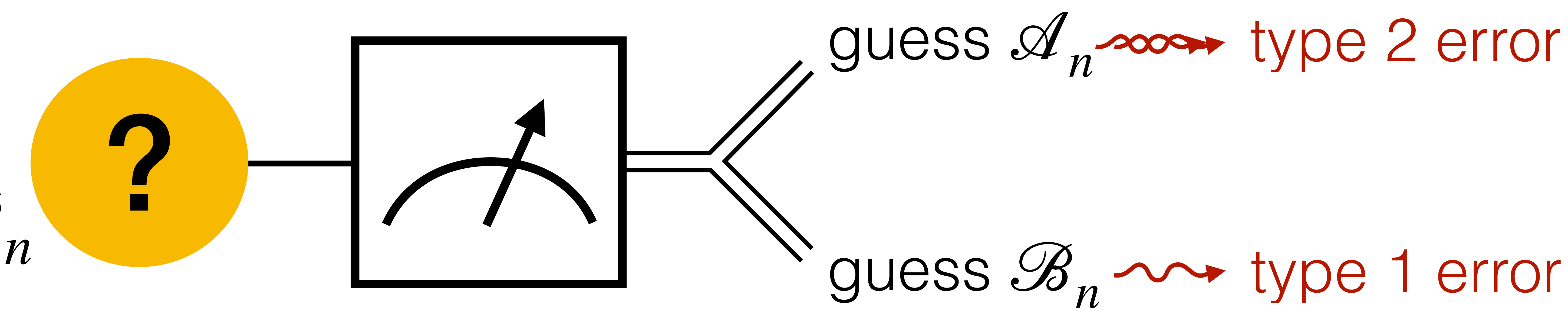

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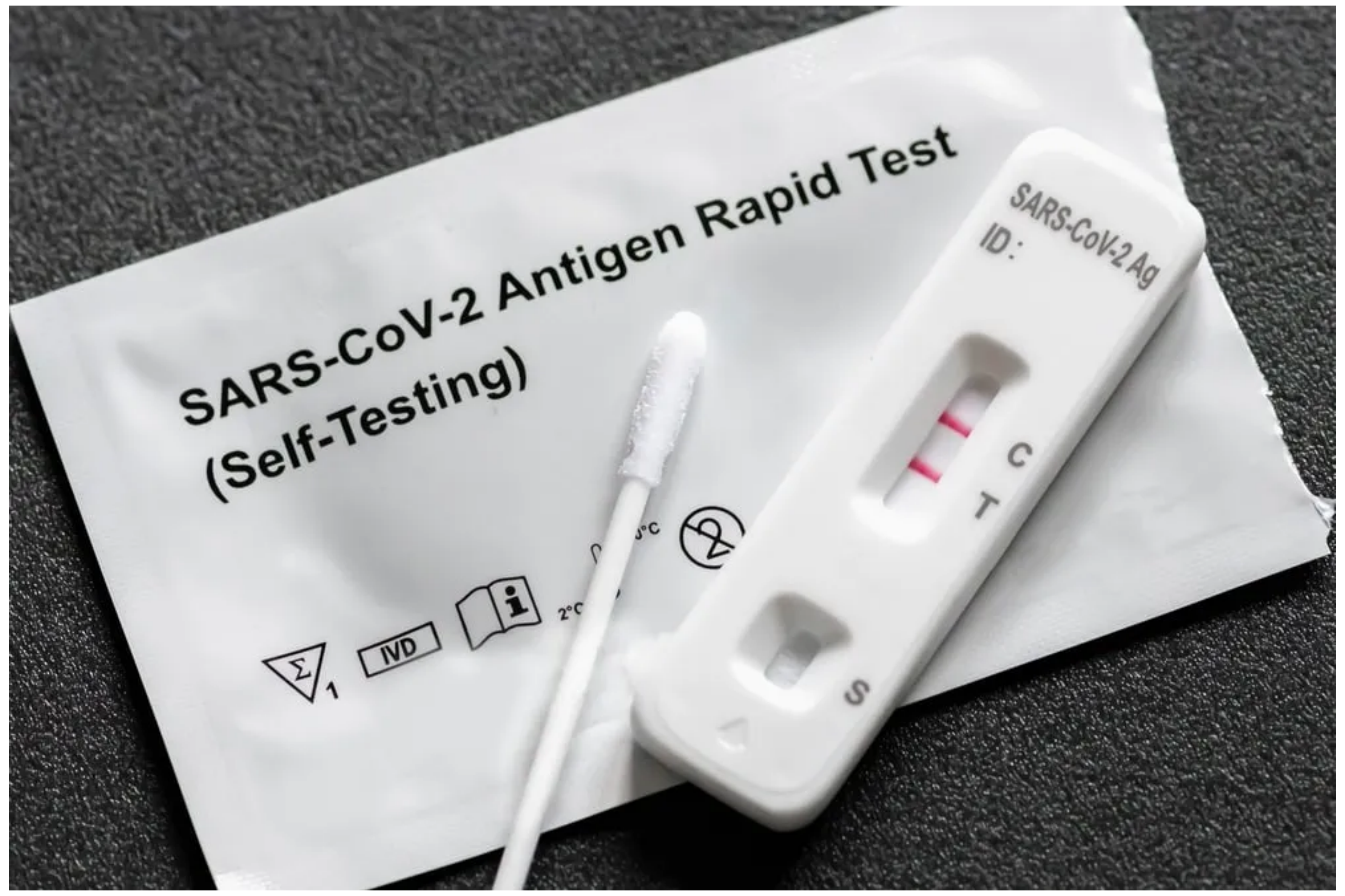

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Single-letter expressions (= with no limits $\varepsilon \to 0$ or $n \to \infty$): sometimes, but unrealistic in most cases…

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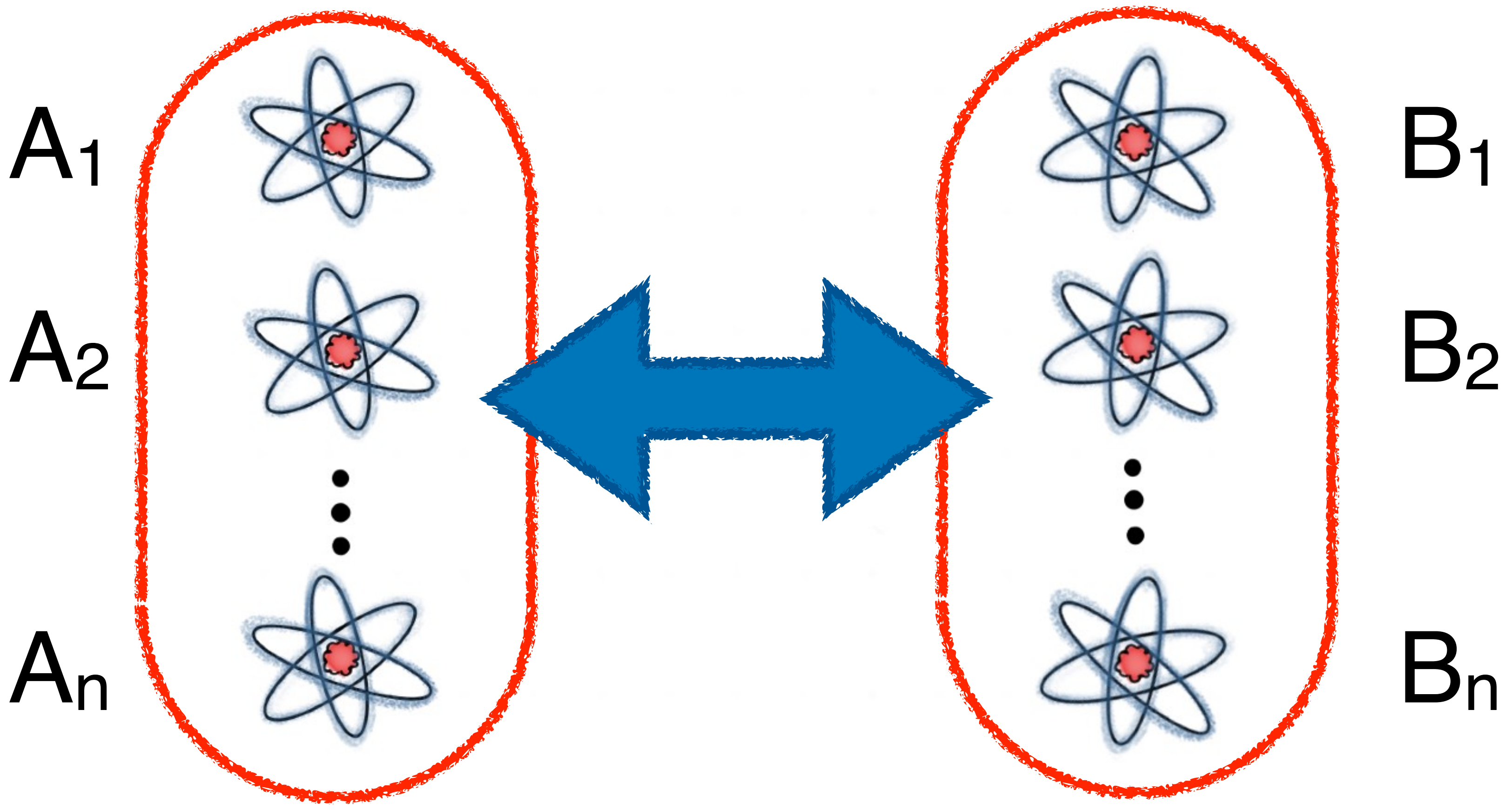
- More complicated sets…

Example of more complicated sets:

Separable states (SEP)



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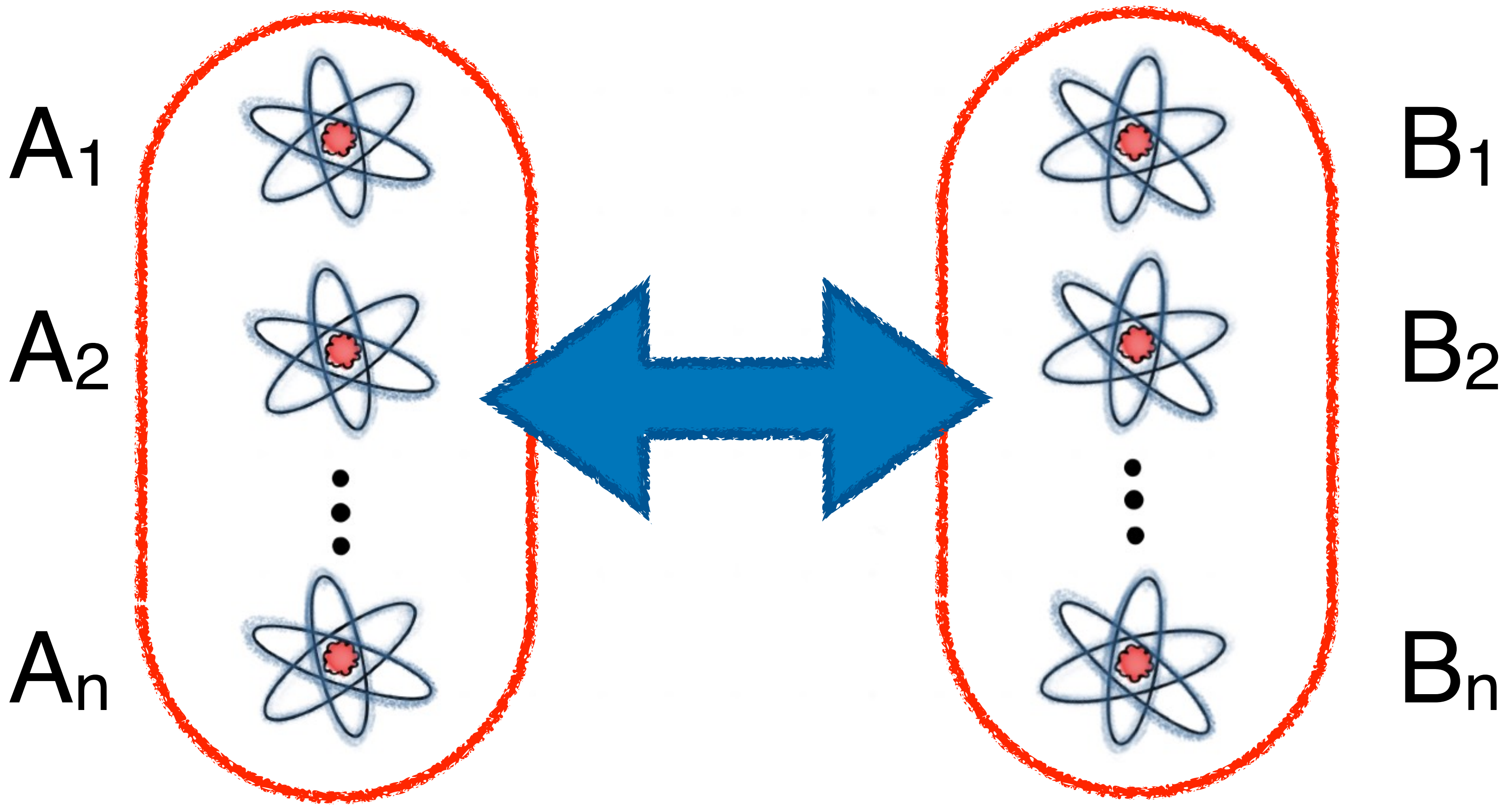

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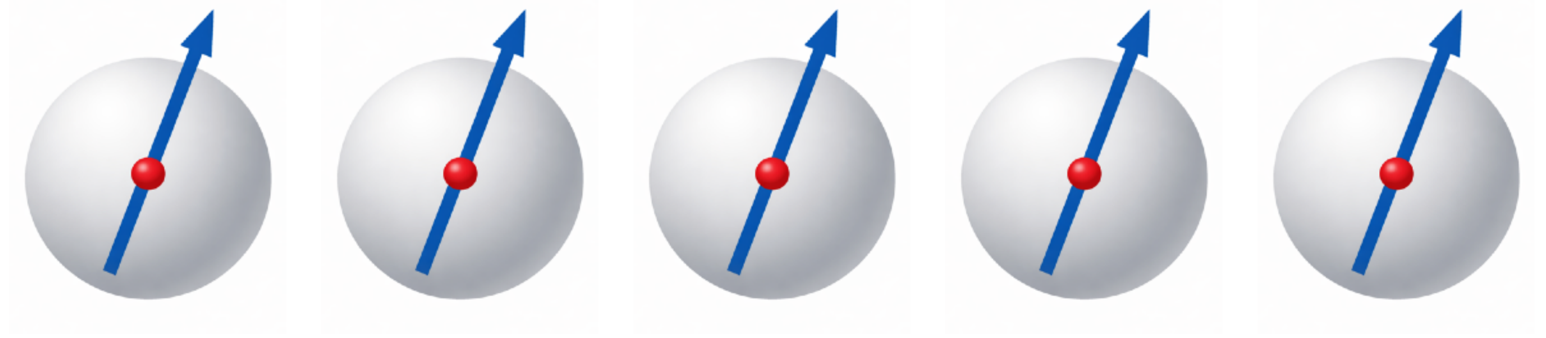
Stabiliser states (STAB)

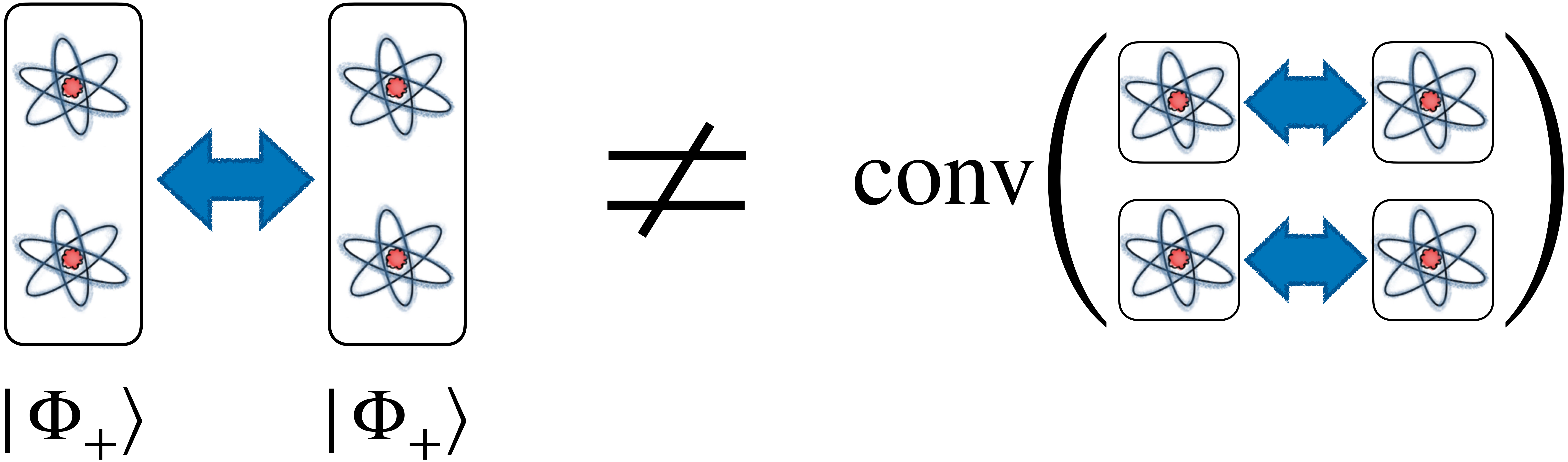


n-qubit system

Clifford group

$$\mathrm{STAB}_n := \mathrm{conv}\left\{ C|0\rangle\langle 0|^{\otimes n} C^\dagger : \ C \in \mathscr{C}_n \right\}$$

Genuinely correlated: extremal points are <u>not</u> product across the copies.



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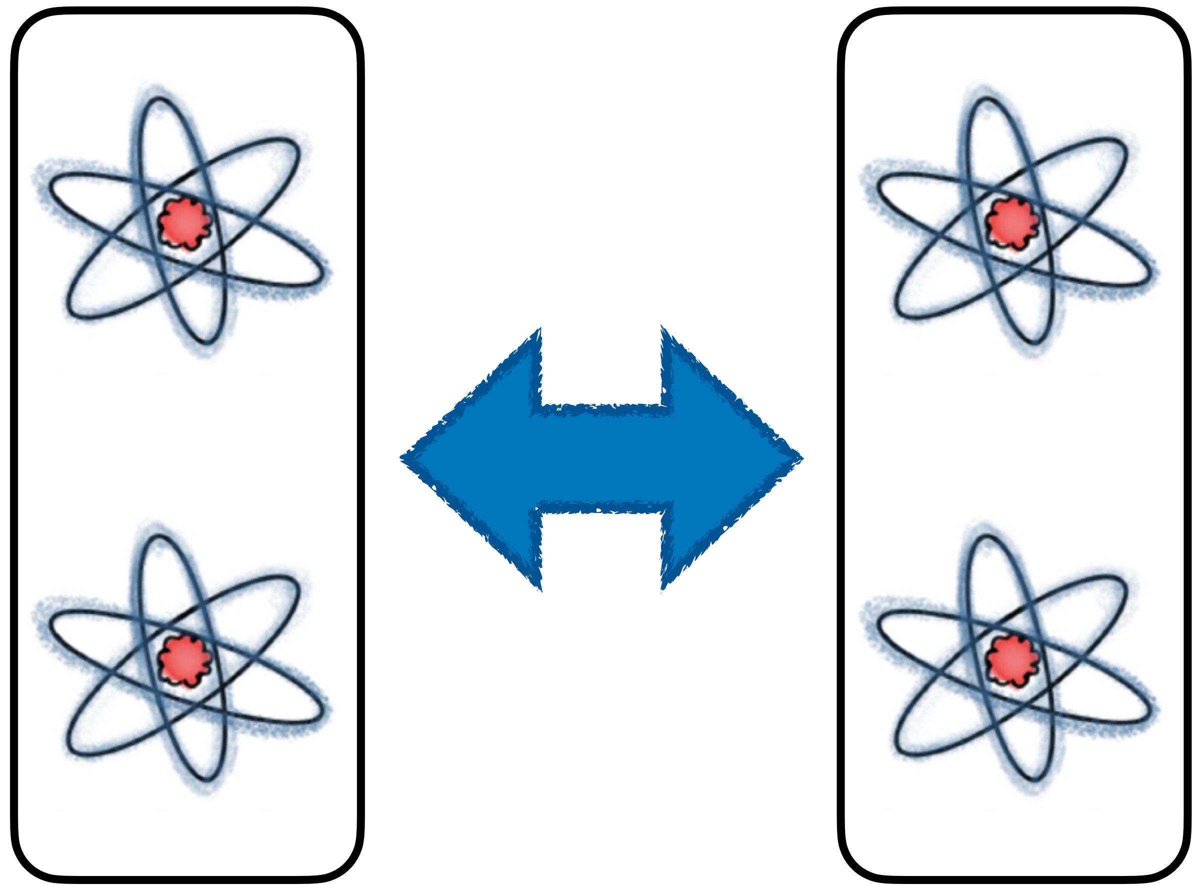
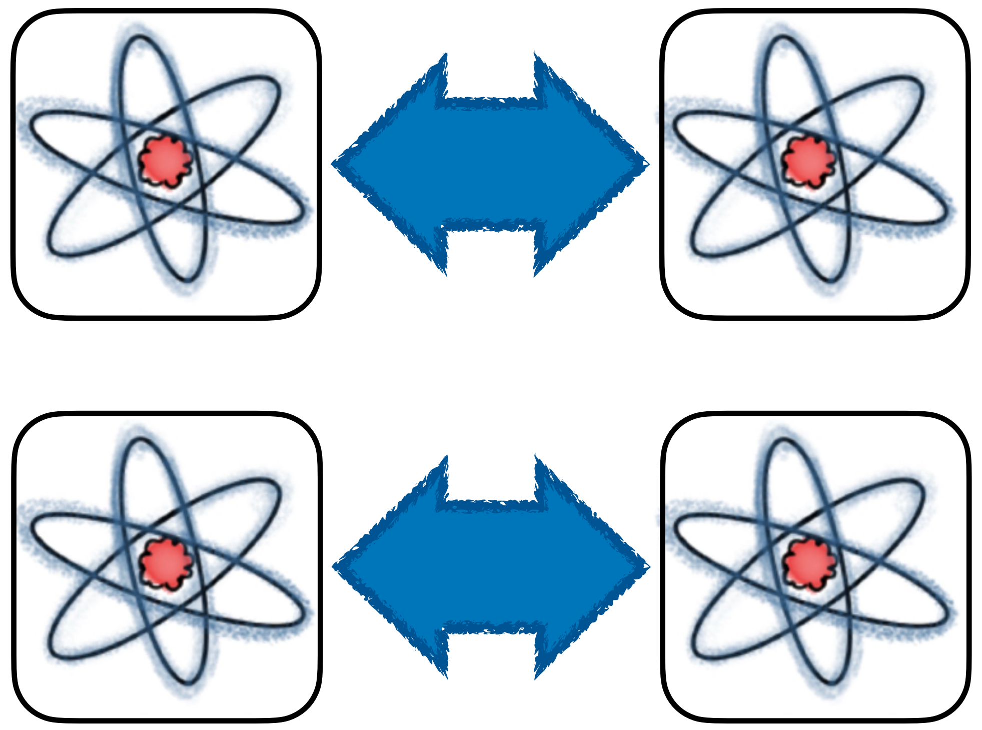


$|\Phi_+\rangle \qquad |\Phi_+\rangle$

$$\neq \quad \mathrm{conv}\Big(\Big)$$



$$\sum_i p_i \; \sigma^{(i)}_{A_1:B_1} \otimes \sigma^{(i)}_{A_2:B_2}, \qquad \sigma^{(i)}_{A_j:B_j} \in \mathrm{SEP}_1$$

Some prior results:

- Quantum Stein's lemma → simple IID vs simple IID:

$$\mathrm{Stein}\Big((\rho^{\otimes n})_n \,\|\, (\sigma^{\otimes n})_n\Big) = D(\rho\|\sigma)$$

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[Bjelaković et al. 2005]

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[Nötzel 2014]

Single-letter!

Note: same formula, but $\mathscr{A}_1 \mapsto \mathrm{conv}(\mathscr{A}_1)$. Coincidence?

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- Generalised quantum Stein's lemma → simple IID vs SEP/STAB

$$\mathrm{Stein}\big((\rho^{\otimes n})_n \,\|\, \mathrm{SEP}\big) = D^{\infty}(\rho\|\mathrm{SEP})$$

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[L./Berta/Regula 2026]

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[L./Berta/Regula 2026]

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Last single-letter formula you will see in this talk.

Result	Null hypothesis			Alternative hypothesis		
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Q. Sanov theorem [Bjelakovič et al. 2005], [Noetzel 2014]	Y	N	N	N	N	N
Doubly composite IID [Berta/Brandão/Hirche 2017]	Y	N	N	Y	N	N
Generalised q. Stein's lemma [Hayashi/Yamasaki 2025]	N	N	N	Y	Y	Y
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Generalised q. Sanov theorem [L./Berta/Regula 2026]	Y	Y	Y	N	N	N
[Fang/Fawzi/Fawzi 2025]	Y	Y	N	Y	Y	N
This talk	Y	Y	Y	Y	N	N

From arXiv:2510.06340

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Results

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Proposition. $(\mathscr{A}_n)_n$ arbitrary sequence, with $\mathscr{A}_n$ convex & closed under permutations; $\mathscr{B}_1$ closed. Then

$$\mathrm{Stein}\big(\mathscr{A} \,\|\, \mathscr{B}_1^{\mathrm{av}}\big) = \mathrm{Stein}\Big(\mathscr{A} \,\|\, \big(\mathrm{conv}(\mathscr{B}_1)\big)^{\mathrm{iid}}\Big)$$

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(And same for regularised relative entropy.)

$\Longrightarrow$ Composite IID gets us AV for free (in the alternative hypothesis).

Proof idea. Uniform combination of *k* copies of ρ_1 and *n-k* copies of ρ_2:

$$\Omega_n := \binom{n}{k}^{-1} \sum_{I \subseteq [n]:\, |I|=k} \rho_1^{\otimes I} \otimes \rho_2^{\otimes I^c}$$

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(+ discretisation procedure…)

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Theorem (Generalised quantum Sanov theorem revisited). *Null hypothesis* $\mathcal{A} = \mathrm{SEP}$ *or* $\mathcal{A} = \mathrm{STAB}$*;* $\mathcal{B}_1$ *closed set of states. Then*

$$\mathrm{Stein}\big(\mathcal{A}\,\big\|\,\mathcal{B}_1^{\mathrm{iid}}\big) = \min_{\mu\in\mathcal{P}(\mathcal{B}_1)} \lim_{n\to\infty} \frac{1}{n} \inf_{\rho_n\in\mathcal{A}_n} D\left(\rho_n \,\middle\|\, \int_{\mathcal{B}_1} \mathrm{d}\mu(\sigma_1)\, \sigma_1^{\otimes n}\right),$$

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Best we can hope for, even when $\mathscr{A}_n = \{\rho^{\otimes n}\}$ and $\mathscr{B}_1 = \{\sigma_1, \sigma_2\}$!

[Mosonyi/Szilágyi/Weiner 2022]

Next: what happens to the generalised quantum Sanov theorem if the alternative hypothesis is composite IID (or AV)?

Theorem (Generalised quantum Sanov theorem revisited). *Null hypothesis* $\mathcal{A} = \mathrm{SEP}$ *or* $\mathcal{A} = \mathrm{STAB}$*;* $\mathcal{B}_1$ *closed set of states. Then*

$$\mathrm{Stein}\big(\mathcal{A}\,\big\|\,\mathcal{B}_1^{\mathrm{iid}}\big) = \min_{\mu\in\mathcal{P}(\mathcal{B}_1)} \lim_{n\to\infty} \frac{1}{n} \inf_{\rho_n\in\mathcal{A}_n} D\left(\rho_n \,\middle\|\, \int_{\mathcal{B}_1} \mathrm{d}\mu(\sigma_1)\, \sigma_1^{\otimes n}\right),$$

where $\mathcal{P}(\mathcal{B}_1)$ *denotes the set of all probability measures on* $\mathcal{B}_1$*.*

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Best we can hope for, even when $\mathscr{A}_n = \{\rho^{\otimes n}\}$ and $\mathscr{B}_1 = \{\sigma_1, \sigma_2\}$! [Mosonyi/Szilágyi/Weiner 2022]

⟶ improved formula for composite IID vs composite IID/AV! [Berta/Brandão/Hirche 2021]

Proof technique:

1. measure with the “right” measurement (chosen via minimax);
2. lift classical result for achievability;
3. weak converse by relaxing hypothesis testing relative entropy to Umegaki.

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Classical setting. Sequence $(\mathscr{F}_n)_n$ of sets of *n*-symbol probability distributions.

Assume that:

- $\exists$ full-support distribution $P_0 \in \mathscr{F}_1$.
- $\mathscr{F}_n$ closed under P_0-depolarising channel action
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(replace each symbol with one drawn from P_0 with probability δ, independently)

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$$\Pr_{x^n \sim Q_n}\{x^n \text{ has type } V\} \leq \exp\left[-D(V^{\otimes n} \| \mathscr{F}_n) + o(n)\right]$$

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$\Longrightarrow$ nice classical Chernoff-Stein lemma that we can then lift to quantum!

$\Longrightarrow$ Classical constrained de Finetti reduction theorem: if $Q_n \in \mathscr{F}_n$ perm. symm.,

$$Q_n \leq \int \mathrm{d}P \, \exp\left[-D(P^{\otimes n} \| \mathscr{F}_n) + o(n)\right] \, P^{\otimes n}$$

see also [Duan/Severini/Winter 2016],
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Conjecture. For all permutationally symmetric separable states $\sigma_{A^nB^n}$,

$$\sigma_{A^nB^n} \leq \int \mathrm{d}\omega_{AB} \, \exp\left[-nD^{\infty}(\omega \| \mathrm{SEP}) + o(n)\right] \omega_{AB}^{\otimes n}$$

Summary

- Regularised formulas for Stein exponents of very general hypotheses vs composite IID/AV
- $\Longrightarrow$ Generalised quantum Sanov theorem "revisited"
- Alternative hypothesis: AV = composite IID + convex hull
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Summary

- Regularised formulas for Stein exponents of very general hypotheses vs composite IID/AV
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Thank you!