A Resource Theory Of Asymmetry For Gaussian Bosonic Systems

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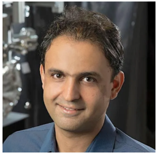

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Outline

Introduction & Background

Gaussianity & Symmetries As Operational Constraints

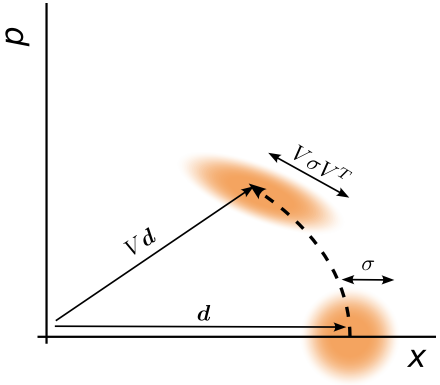
- Bosonic systems: photonic quantum computing [1], quantum networks [2], etc.
- Operational constraints on bosonic dynamics:

  Gaussianity
  - low order approximation of quantum correlations [3]
  - efficient to prepare and manipulate [4]

  Symmetries
  - fundamental restrictions on dynamics
  - conservation laws [5]

Gaussianity

- **Bosonic modes**: $\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n$
  $\rightarrow \hat{a}_j = (\hat{x}_j + \hat{p}_j)/\sqrt{2}$
- **Gaussian states**: $\hat{\rho} \in \mathcal{G}$
  (fully characterized by $\boldsymbol{d}$, σ)
- **Gaussian channels**: CPTP $\mathcal{E} : \mathcal{G} \rightarrow \mathcal{G}$
- **Gaussian unitaries**: $\hat{D}_{\boldsymbol{r}} \hat{V}\ , \quad V \Omega V^{\mathrm{T}} = \Omega$
  $\rightarrow$ **symplectic** if $\boldsymbol{r} = \boldsymbol{0}$ (see diagram)
  $\rightarrow$ **passive** if also $V V^{\mathrm{T}} = I$



Symmetries

- **Representation** of symmetry group G:

  $g \mapsto \hat{S}(g)$ for all $g \in G$

- **Invariant** states & **Covariant** channels:

  $$\hat{\rho} = \hat{S}(g)\ \hat{\rho}\ \hat{S}(g)^\dagger$$

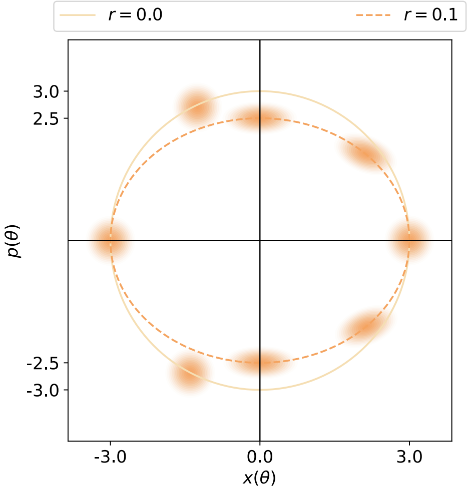
  $$\mathcal{E}(\cdot) = \hat{S}(g)^\dagger\ \mathcal{E}\Big(\hat{S}(g)(\cdot)\hat{S}(g)^\dagger\Big)\ \hat{S}(g)$$

- Invariance on phase space:

  $$S(g)\boldsymbol{d} = \boldsymbol{d}$$

  $$S(g)\sigma S(g)^{\mathrm{T}} = \sigma$$

- **Conjugate representation**: $g \mapsto \hat{S}(g)^*$



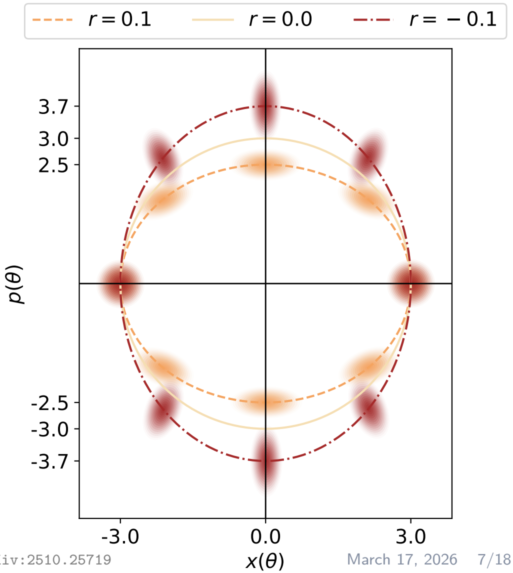
Gaussianity & Symmetries

What representations
(i) **admit invariant states**, and
(ii) **preserve Gaussianity**?

Theorem

Every such representation
(i) is symplectic,
(ii) is equivalent to a passive representation, and
(iii) admits invariant Gaussian ***pure*** *states.*

We also fully characterize their irreps.



Examples

- $\mathrm{U}(1)$ symmetry: phase insensitivity, energy / photon conservation, angular momentum in z direction

$$t \mapsto \exp\left(it \sum_{j=1}^{n} \omega_j \hat{a}_j^{\dagger} \hat{a}_j\right), \quad \omega_j \in \mathbb{R}$$

$\omega_j < 0$ can be implemented thermodynamically via modulation.

$\rightarrow$ entangling 2–mode squeezers are allowed on modes with $\omega_j = -\omega_k$

- $\mathrm{SU}(2)$ symmetry: Schwinger boson representation of angular momentum [6]

$$U \mapsto \hat{S}(U) = e^{-i\alpha_1 \hat{J}_z} e^{-i\alpha_2 \hat{J}_x} e^{-i\alpha_3 \hat{J}_z}, \quad \hat{J}_z = \frac{1}{2}\left(\hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\right), \quad \hat{J}_x = \frac{1}{2}(\hat{a}^{\dagger}\hat{b} + \hat{a}\hat{b}^{\dagger})$$

Examples – side note

- Gaussian channels: X, Y s.t.

$$\boldsymbol{d} \to X\boldsymbol{d}\,, \quad \sigma \to X\sigma X^{\mathrm{T}} + Y\,.$$

- $\mathrm{U}(1)$–covariant Gaussian channels on 1 mode (well-known):

$$X = xI_2\,, \quad Y = yI_2\,,$$
$$\text{s.t. } y \geq |x^2 - 1|\,.$$

- $\mathrm{SU}(2)$–covariant Gaussian channels on 2 modes (our result):

$$X = e^{\phi\Omega}\left(x_+ I_4 + x_- e^{\theta\Omega} X_-\right)\,, \quad Y = yI_4\,,$$
$$\text{where } X_- := (Z \oplus Z)\, V_{\mathrm{bs}}\left(\frac{\pi}{2}\right)\,,$$
$$\text{s.t. } y \geq \sqrt{\left((x_+ - 1)^2 + x_-^2\right)\left((x_+ + 1)^2 + x_-^2\right)}\,.$$

Main Questions

Under Gaussian & symmetric operational constraints, we identify:

- "Free" channels = Gaussian covariant channels
- "Free" states = Gaussian invariant states

Questions

1. How can we realize free channels?
   Spoiler: We construct free dilations.

2. How can we quantify "resource"?
   Spoiler: We derive monotones under free channels.

Main results

Covariant Gaussian Dilation

Lemma (Invariant Gaussian Purifications)

Every Gaussian state $\hat{\rho}_A$ on system A admits a Gaussian purification $|\psi\rangle_{A\overline{A}}$ [7], which is invariant if we choose the representation on $\overline{A}$ as $\hat{S}_{\overline{A}} = \hat{S}^*_A$.
All such purifications are equivalent up to an invariant Gaussian unitary $\hat{U}_{\overline{A}}$ on $\overline{A}$.

How can we realize covariant Gaussian channels?
Covariant (qudit) dilation [8] ✓   Gaussian dilation [9] ✓   Combined **?**

Theorem (Covariant Gaussian dilation)

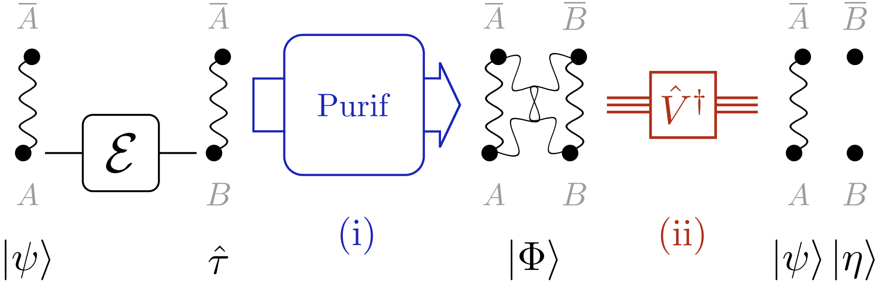
Every covariant Gaussian channel $\mathcal{E} : A \to B$ admits a covariant Gaussian dilation

$$\mathcal{E}(\cdot) = \mathrm{tr}_{A\overline{B}} \left[\hat{V}_{AB\overline{B}} \left((\cdot) \otimes |\eta\rangle\langle\eta|_{B\overline{B}} \right) \hat{V}^\dagger_{AB\overline{B}} \right] ,$$

for invariant Gaussian $|\eta\rangle$ and $\hat{V}$, and we choose the representation on $\overline{B}$ as $\hat{S}_{\overline{B}} = \hat{S}^*_B$.

Covariant Gaussian Dilation

- Novel proof of Stinespring dilation for covariant Gaussian channels:
  (i) Purify channel output (ii) Decouple purification



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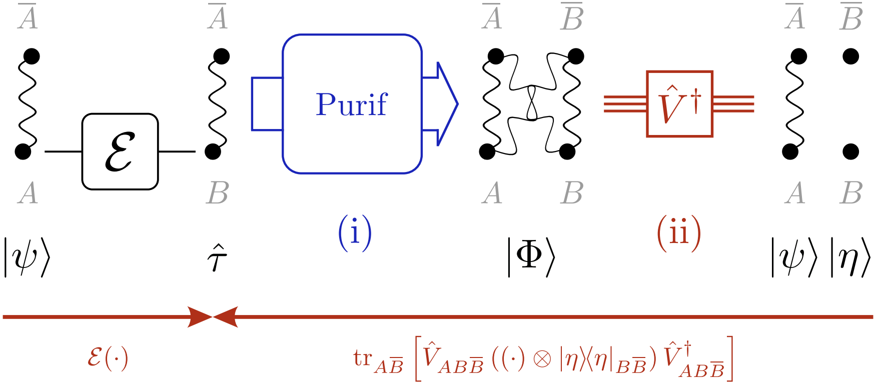

Gaussian Asymmetry Monotones

- Type–1 asymmetry: $S(g)\boldsymbol{d} \neq \boldsymbol{d}$ for some $g \in G$.
- Type–2 asymmetry: $S(g)\sigma S(g)^{\mathrm{T}} \neq \sigma$ for some $g \in G$

Theorem

$2 \not\to 1$. *No covariant channel can convert type–2 to type–1 asymmetry.*

$1 \to 2$. *There always exists a non-Gaussian covariant channel that converts type–1 to type–2 asymmetry, .*

Remark: For non-Gaussian states, type–m asymmetry becomes relevant.

Gaussian Asymmetry Monotones

- **Type–1 asymmetry monotones.**

$$f_\mu(\hat{\rho}) = \text{ rank of } \boldsymbol{d} \text{ projected onto non-trivial irrep } \mu\,.$$

- Example: For Abelian symmetries such as $\mathrm{U}(1)$, $f_\mu(\hat{\rho}) \in \{0, 1\}$.

Gaussian Asymmetry Monotones

- **Type–2 asymmetry monotones.** A recipe:
  1. Pick Gaussian state $\hat{\rho}$ & Remove first moment $\hat{\rho} \to \mathcal{D}(\hat{\rho})$
  2. Pick asymmetry monotone f & Calculate $f(\mathcal{D}(\hat{\rho}))$
- Some monotones take closed forms for Gaussian states in terms of $\boldsymbol{d}, \sigma$, e.g.

$$f_{\alpha,g}(\hat{\rho}) \coloneqq D_\alpha(\mathcal{D}(\hat{\rho})||\hat{S}(g)\mathcal{D}(\hat{\rho})\hat{S}(g)^\dagger)\,.$$

- Example: For $\mathrm{U}(1)$ passive representation $t \mapsto \exp(i\omega t I)\,$,

$$\begin{aligned} |r\rangle \mapsto f_{\alpha,\frac{\pi}{2\omega}}(|r\rangle) &= \frac{1}{2}\log\Big(1+\sinh^2(2r)\Big)\ , \\ |\gamma\rangle \mapsto f_{\alpha,t}(|\gamma\rangle) &= 0\,. \end{aligned}$$

Summary & Outlook

Summary

- We study the interplay of Gaussianity & symmetries in bosonic dynamics.
- We fully characterize all symmetry representations that preserve Gaussianity.

Answers

1. How can we realize Gaussian symmetry-respecting channels?
   Answer: We construct Gaussian symmetry-respecting dilations.
2. How can we quantify "resource" in a state?
   Answer: We derive monotones for Gaussian symmetry-respecting channels.

- **Bonus:** Necessary and sufficient conditions for pure Gaussian state convertibility under unitary dynamics (incl. forthcoming 2604.xxxxx with David Jakab).
  Central message: Symmetries + Gaussianity = Fine-grained conservation laws.

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