

Three anagrams of mutual: lautum, umlaut and tumula information

Filippo Girardi

Scuola Normale Superiore, Pisa

Aadil Oufkir¹ Bartosz Regüla²

Marco Tomamichel³ Mario Berta¹ Ludovico Lami⁴



Lukas Schmitt⁵ Laura Burri⁵



SCUOLA
NORMALE
SUPERIORE

¹ RWTH Aachen University, Germany

² RIKEN Center for Quantum Computing, Japan

³ National University of Singapore

⁴ Scuola Normale Superiore, Pisa, Italy

⁵ ETH, Zurich, Switzerland



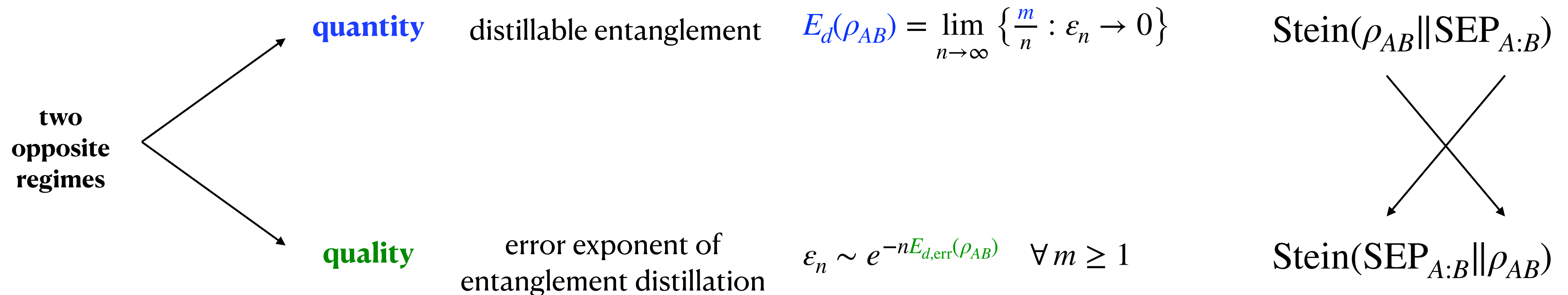
European Research Council

Established by the European Commission

Entanglement manipulation

From quantity to quality

$$\rho_{AB}^{\otimes n} \xrightarrow[\text{non-entangling operations}]{\text{entanglement distillation}} \Lambda^{(n)}(\rho_{AB}^{\otimes n}) \approx_{\varepsilon_n} \Phi_+^{\otimes m} \quad |\Phi_+\rangle := \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$



F. G. S. L. Brandão and M. B. Plenio. *A reversible theory of entanglement and its relation to the second law*. *Comm. Math. Phys.*, 295(3):829–851 (2010)

L. Lami, M. Berta and B. Regula. *Asymptotic quantification of entanglement with a single copy*. *Nat. Phys.* (2026)

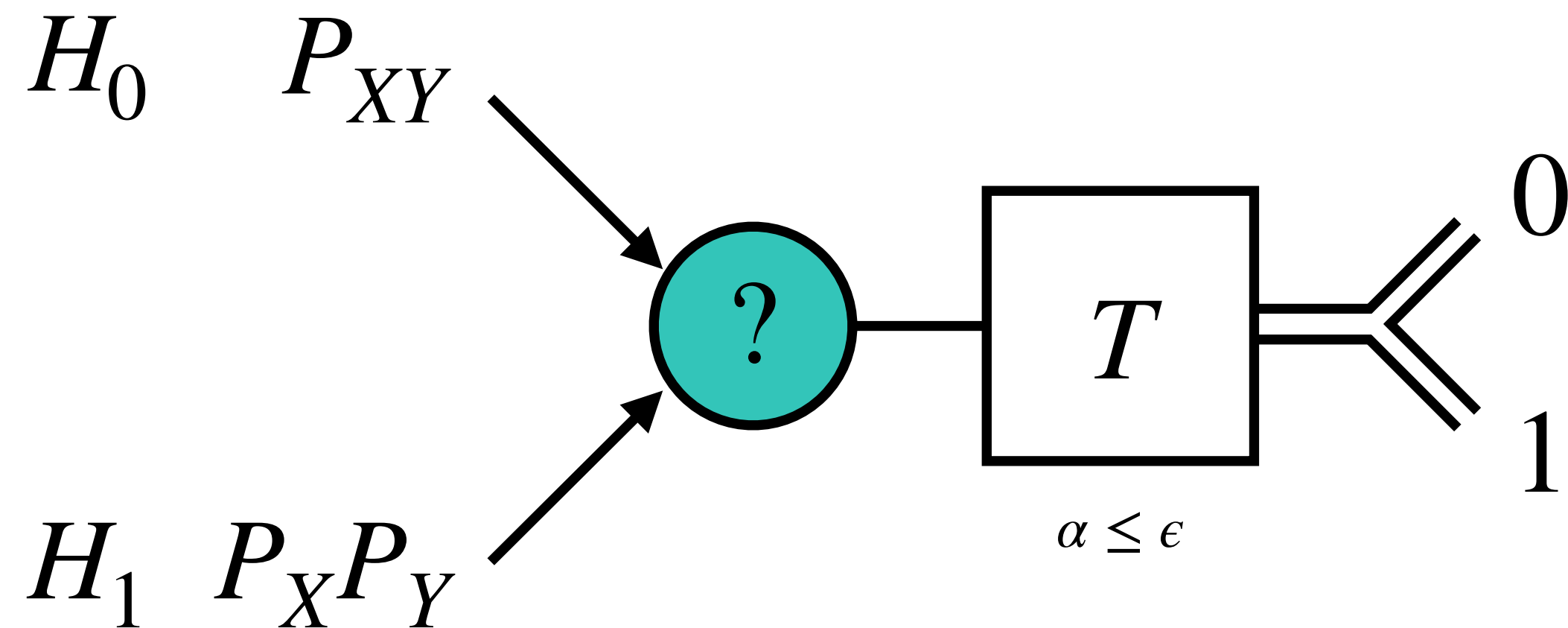
Mutual information

asymmetric
hypothesis testing

classical
communication

$$I(X : Y) := D(P_{XY} \| P_X P_Y)$$

→ Ludovico's and Kun's talks this morning



$$\beta \sim e^{-nD(P_{XY} \| P_X P_Y)} = e^{-nI(X:Y)}$$

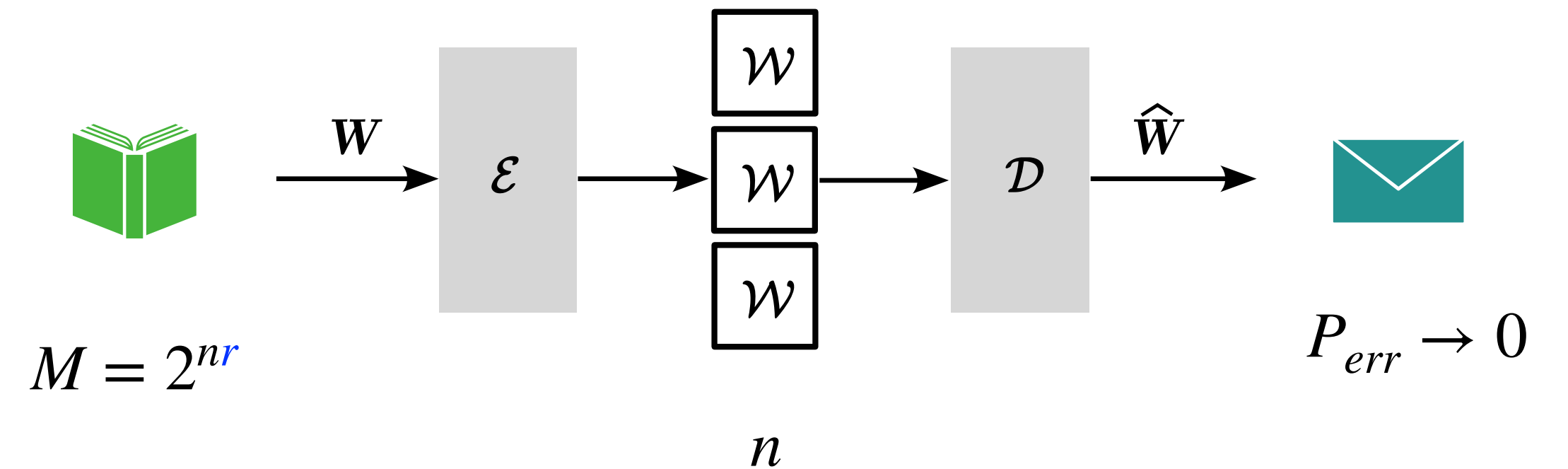
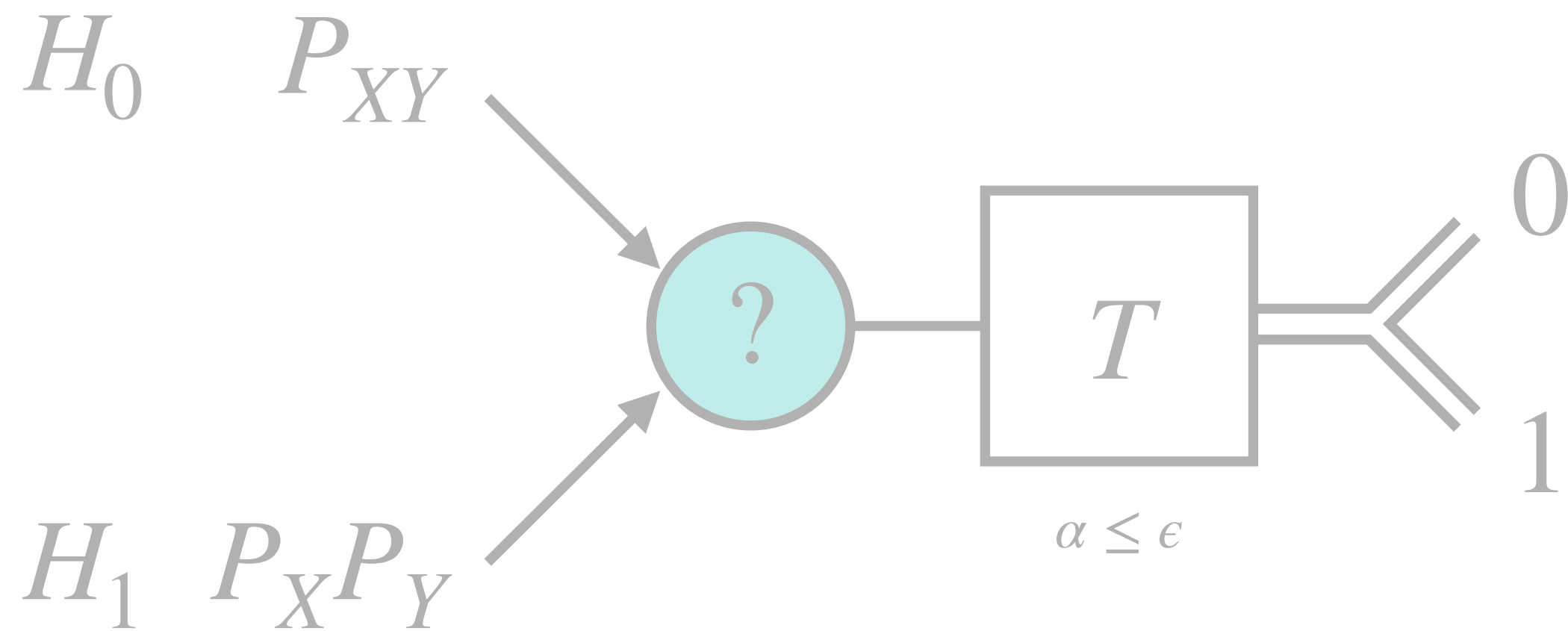
Stein's lemma

Mutual information

asymmetric hypothesis testing

classical communication

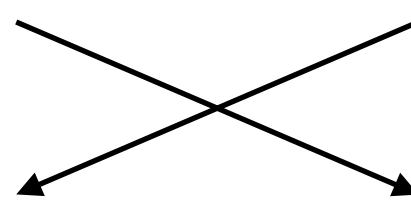
$$I(X : Y) := D(P_{XY} || P_X P_Y)$$



$$C(\mathcal{W}) = \max_{P_X} I(X : Y) \quad P_{XY} = \mathcal{W}_{Y|X} P_X$$

[Shannon, 1948]

How to reverse the mutual information?

MUTUAL

 LAUTUM

$$I(X : Y) := D(P_{XY} \| P_X P_Y) = \min_{Q_Y} D(P_{XY} \| P_X Q_Y) = \min_{Q_X} \min_{Q_Y} D(P_{XY} \| Q_X Q_Y)$$

$$L(X : Y) := D(P_X P_Y \| P_{XY})$$

UMLAUT

$$U(X ; Y) := \min_{Q_Y} D(P_X Q_Y \| P_{XY})$$

TUMULA

$$T(X : Y) := \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| P_{XY})$$

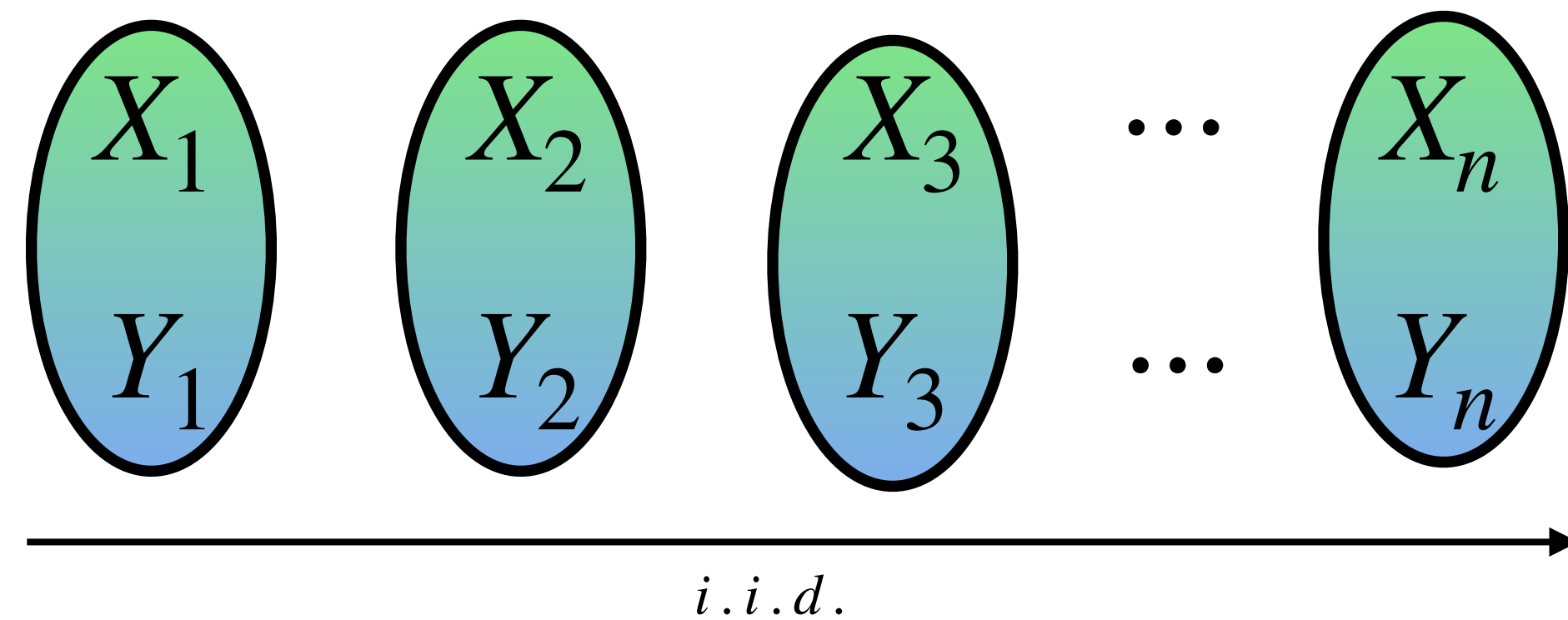
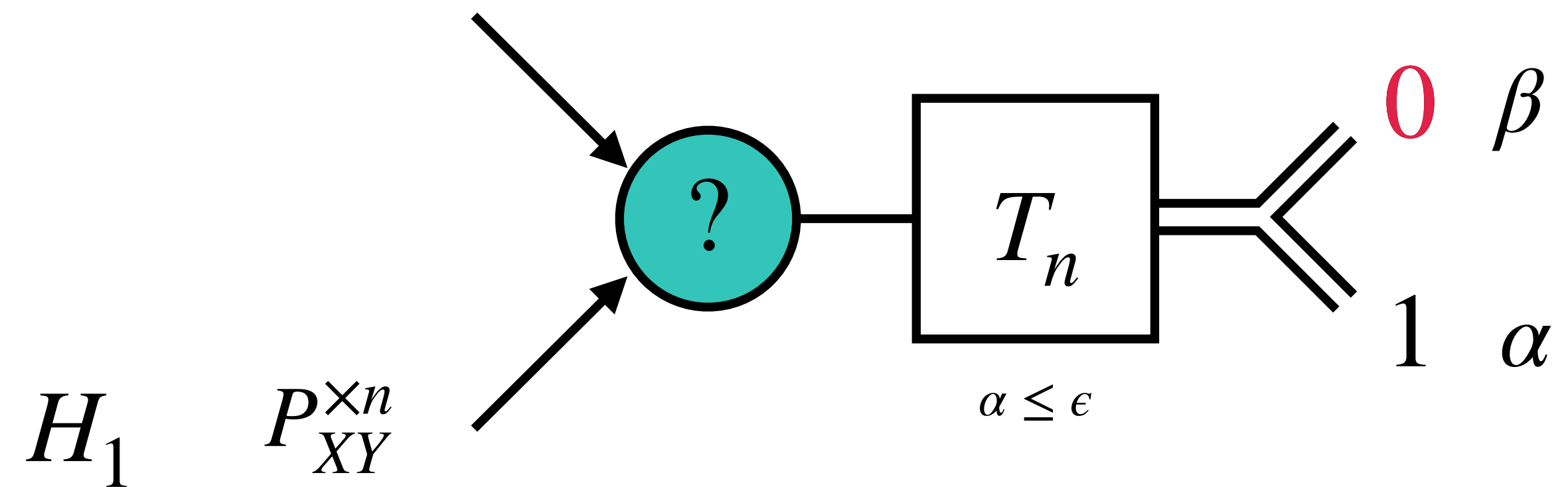
$$P_6(\text{MUTUAL}) \supseteq \{ \text{LAUTUM}, \text{TUMULA}, \text{UMLAUT} \}$$

Asymmetric hypothesis testing

$$L(X : Y) := D(P_X P_Y \| P_{XY})$$

$$U(X ; Y) := \min_{Q_Y} D(P_X Q_Y \| P_{XY})$$

$$T(X : Y) := \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| P_{XY})$$



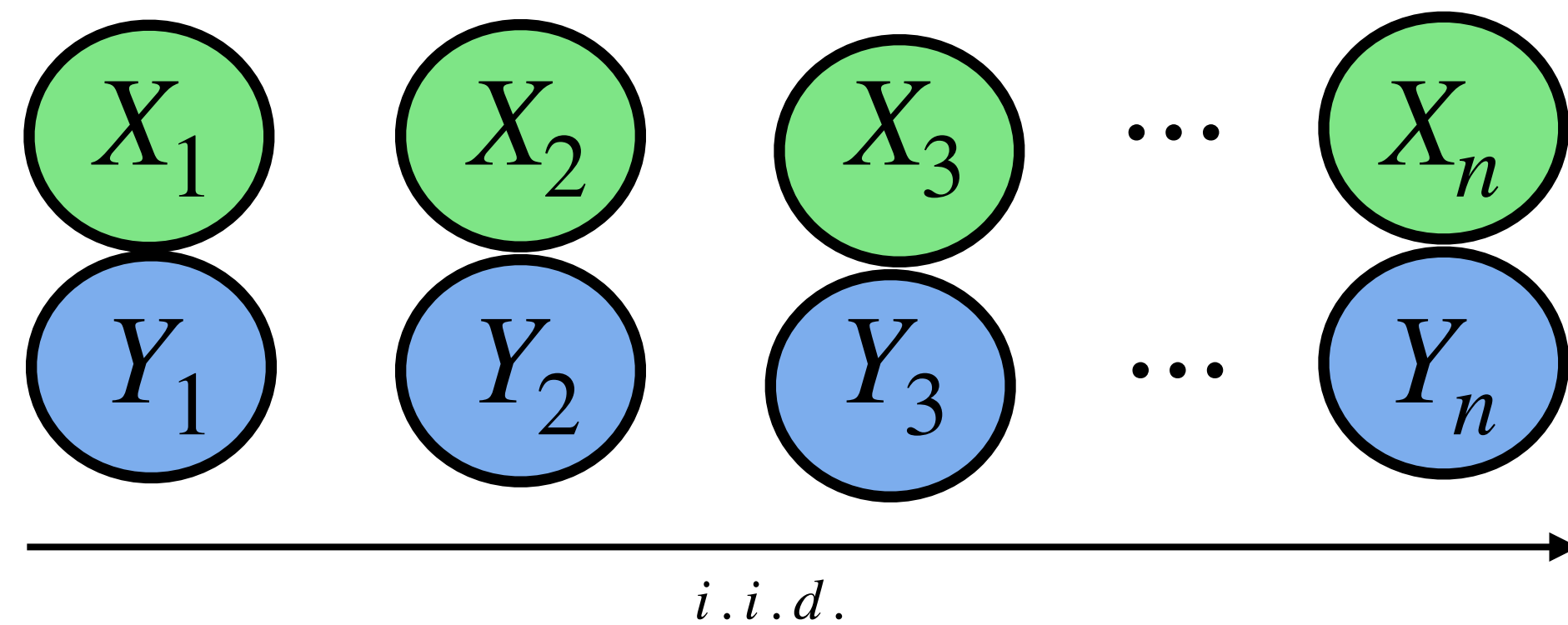
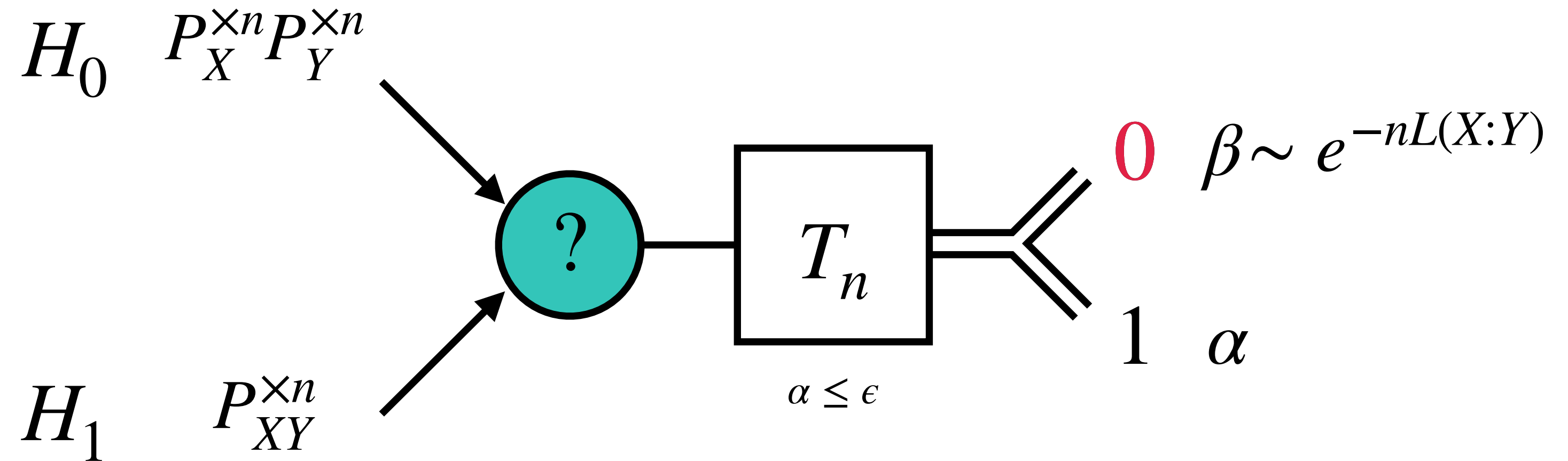
Asymmetric hypothesis testing

$$L(X : Y) := D(P_X P_Y \| P_{XY})$$

$$U(X ; Y) := \min_{Q_Y} D(P_X Q_Y \| P_{XY})$$

$$T(X : Y) := \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| P_{XY})$$

$$L(X : Y) = \text{Stein}(P_X P_Y \| P_{XY})$$



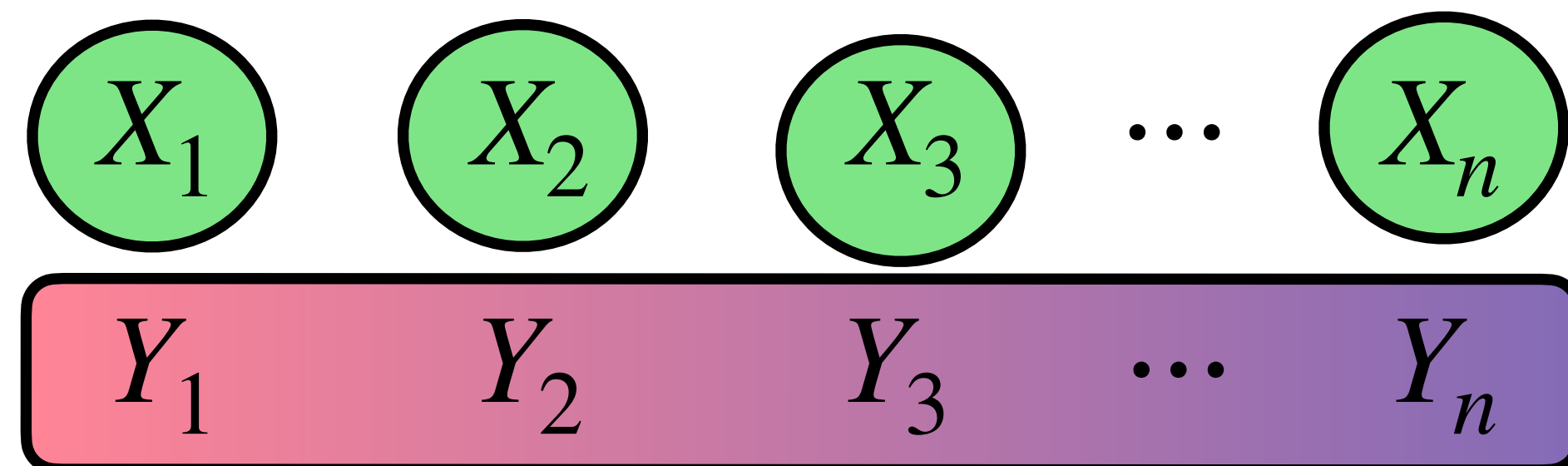
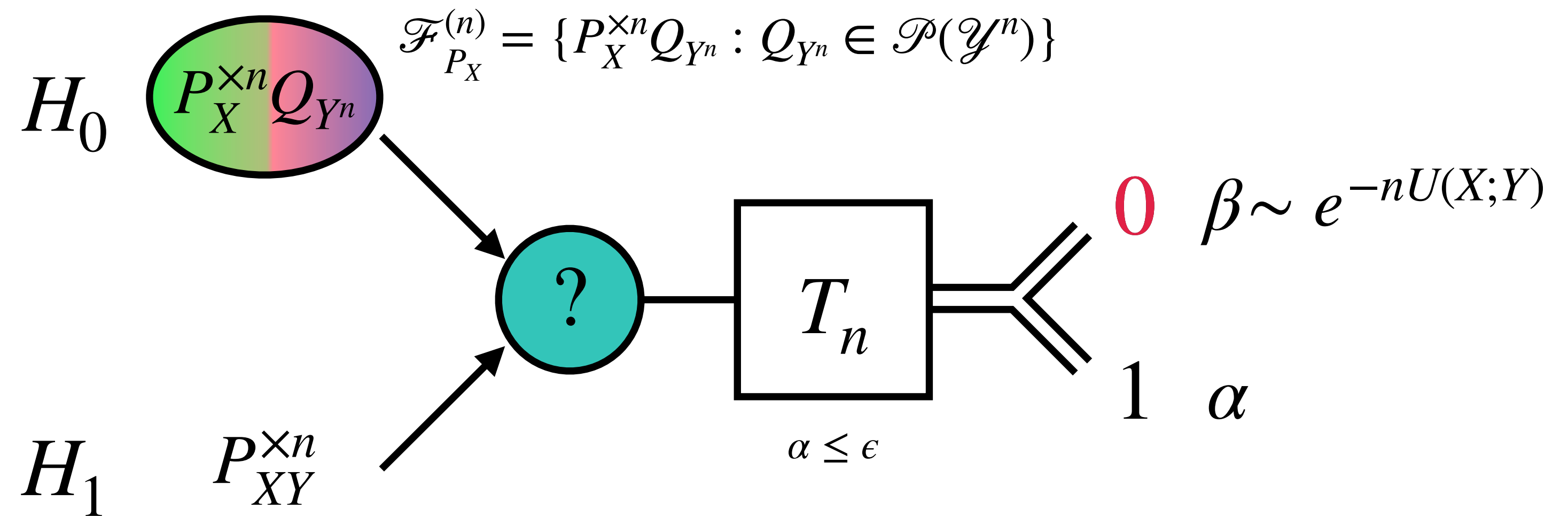
Asymmetric hypothesis testing

$$L(X : Y) := D(P_X P_Y \| P_{XY})$$

$$U(X; Y) := \min_{Q_Y} D(P_X Q_Y \| P_{XY})$$

$$T(X : Y) := \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| P_{XY})$$

$$U(X; Y) = \text{Stein}(\mathcal{F}_{P_X} \| P_{XY})$$



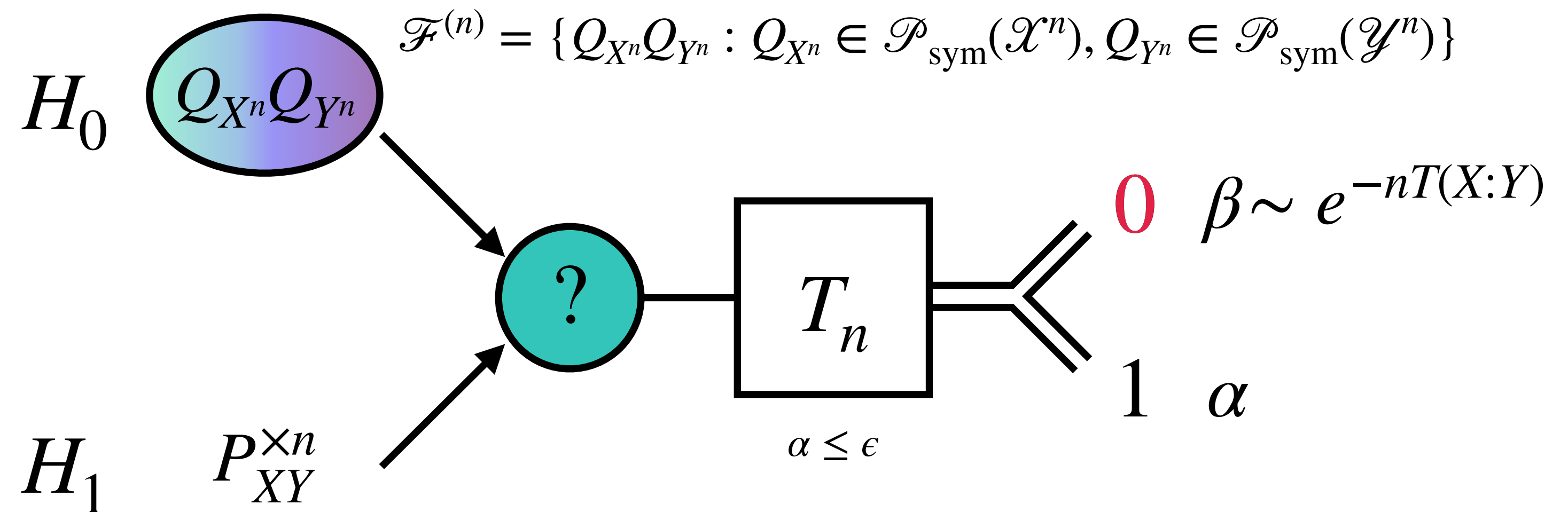
Asymmetric hypothesis testing

$$L(X : Y) := D(P_X P_Y \| P_{XY})$$

$$U(X ; Y) := \min_{Q_Y} D(P_X Q_Y \| P_{XY})$$

$$T(X : Y) := \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| P_{XY})$$

$$T(X : Y) = \text{Stein}(\mathcal{F} \| P_{XY})$$



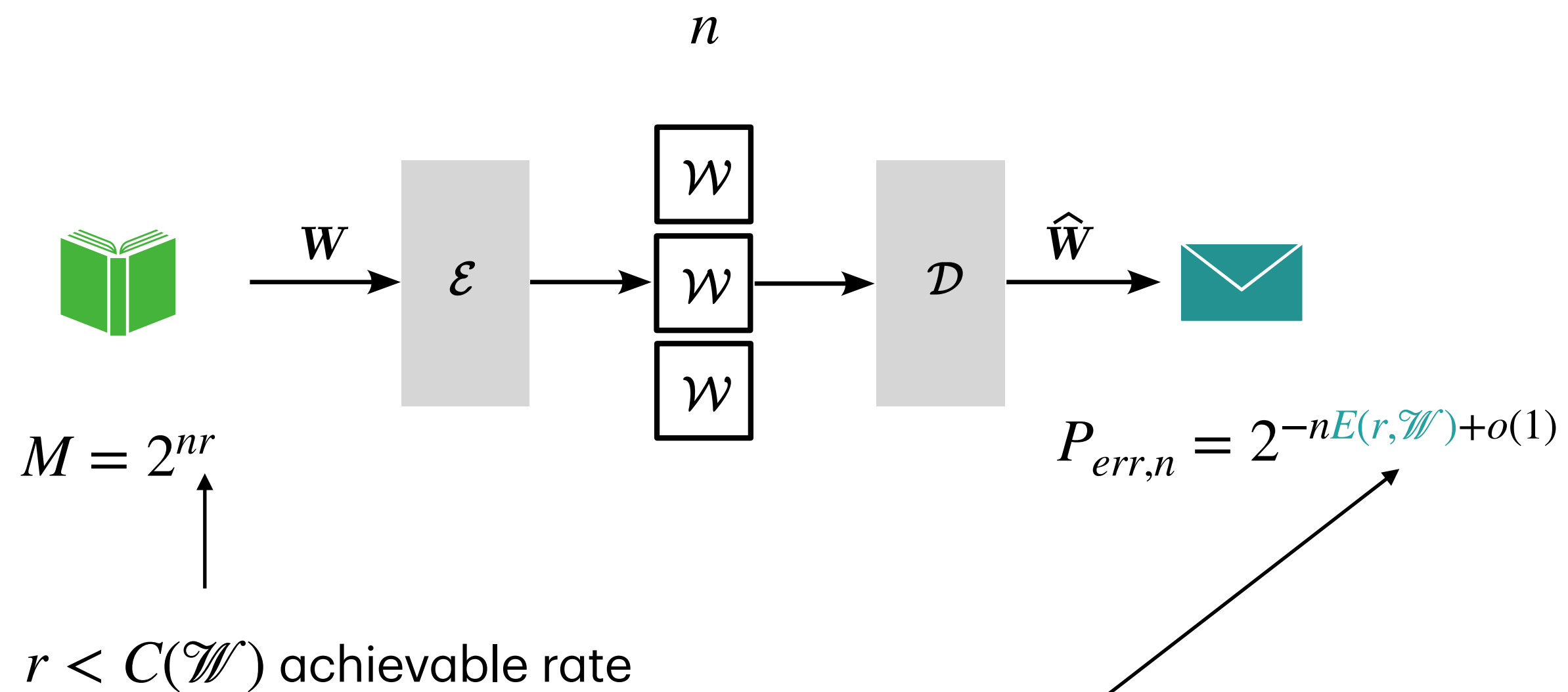
X_1 X_2 X_3 \dots X_n

Y_1 Y_2 Y_3 \dots Y_n

Reliability function

Trading quantity for quality

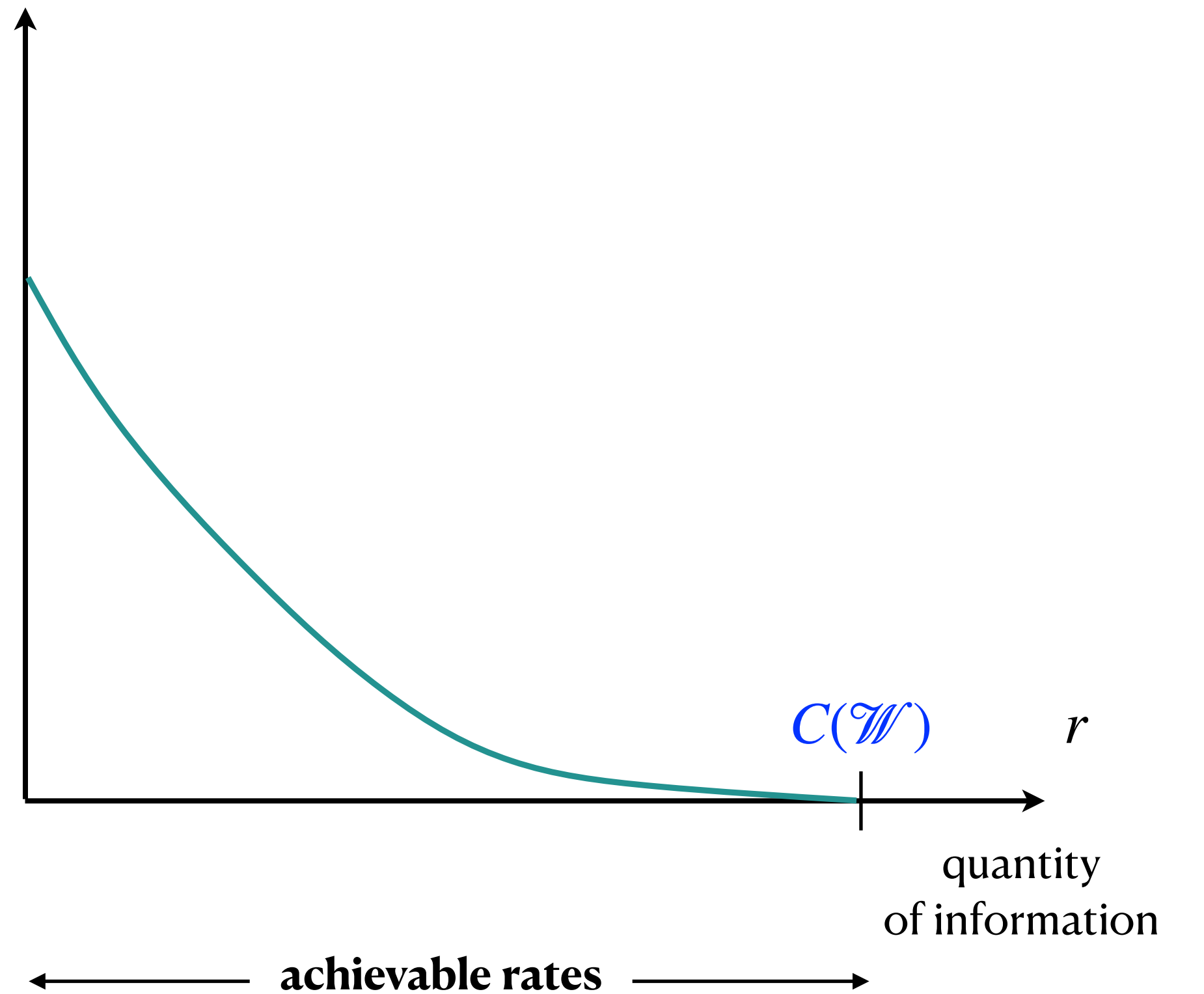
$E(r, \mathcal{W})$ reliability function



reliability function
(or error exponent)

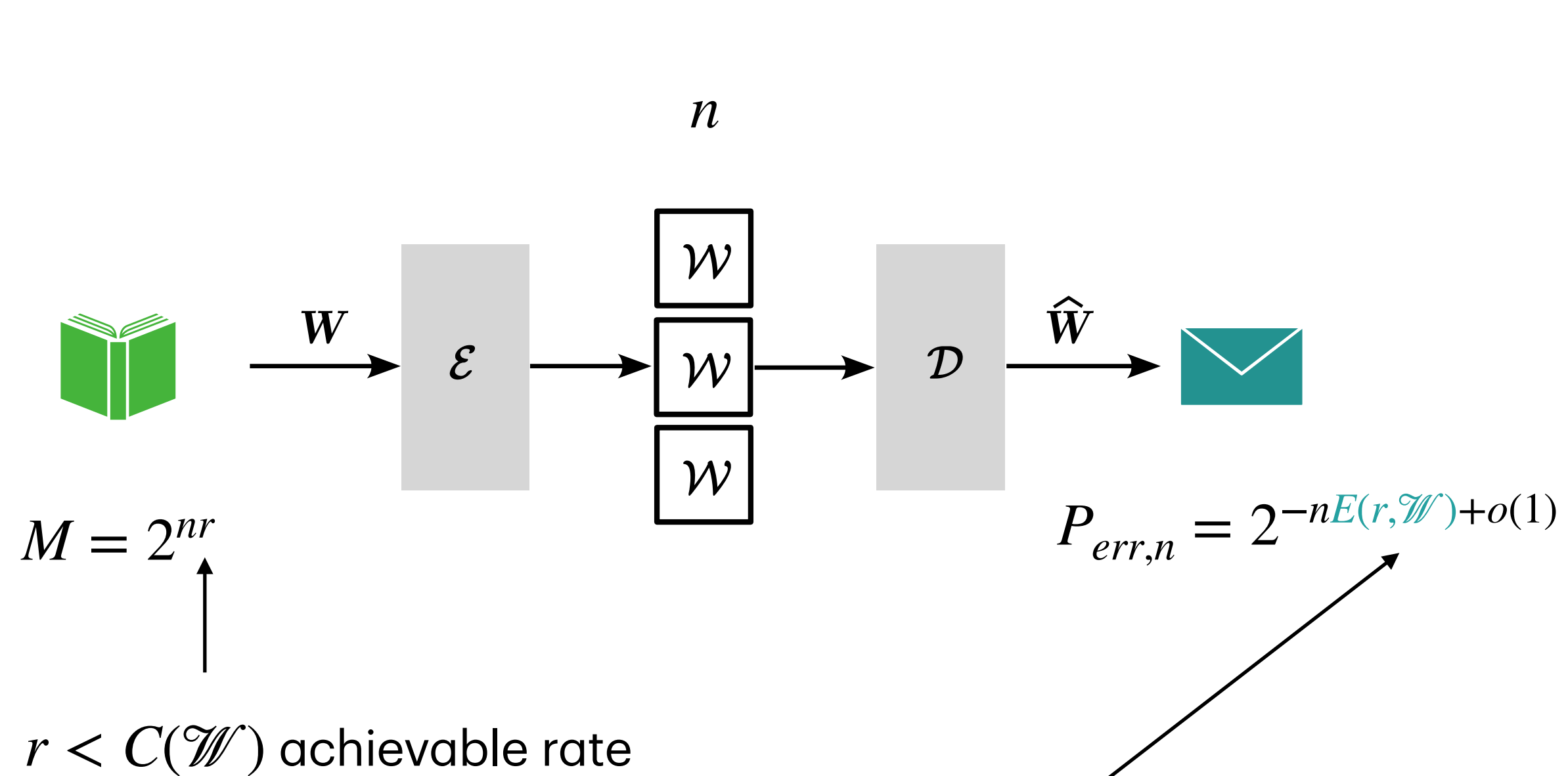
$$E(r, \mathcal{W}) := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \inf_{(\mathcal{E}, \mathcal{D})} P_{err,n}$$

quality
of transmission



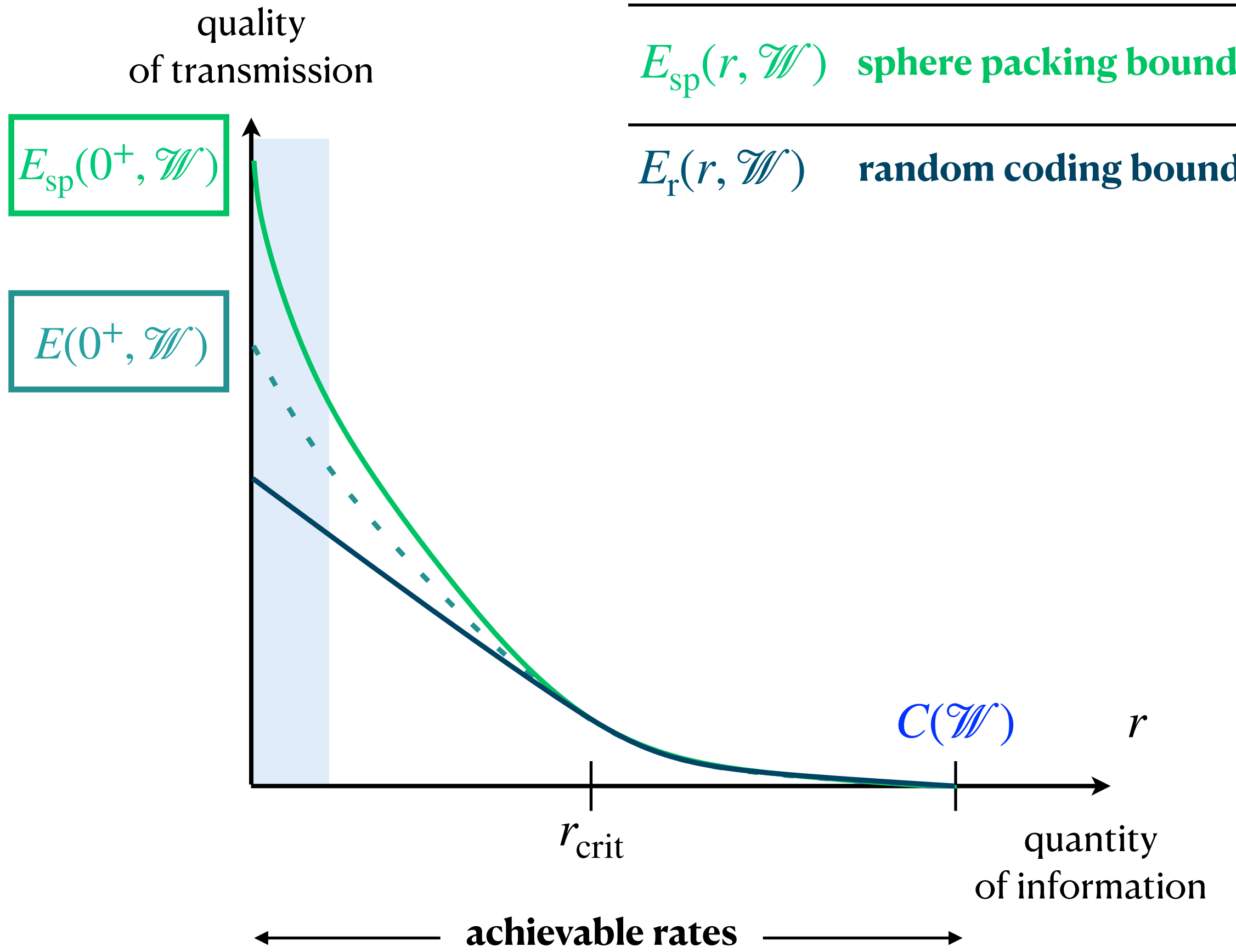
Reliability function

Trading quantity for quality



reliability function
(or error exponent)

$$E(r, \mathcal{W}) := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \inf_{(\mathcal{E}, \mathcal{D})} P_{err,n}$$



$E(r, \mathcal{W})$ reliability function

$E_{sp}(r, \mathcal{W})$ sphere packing bound

$E_r(r, \mathcal{W})$ random coding bound

C. Shannon, R. Gallager, and E. Berlekamp, Information and Control (1967)

R. M. Fano, Transmission of Information: A Statistical Theory of Communication, Wiley, (1961)

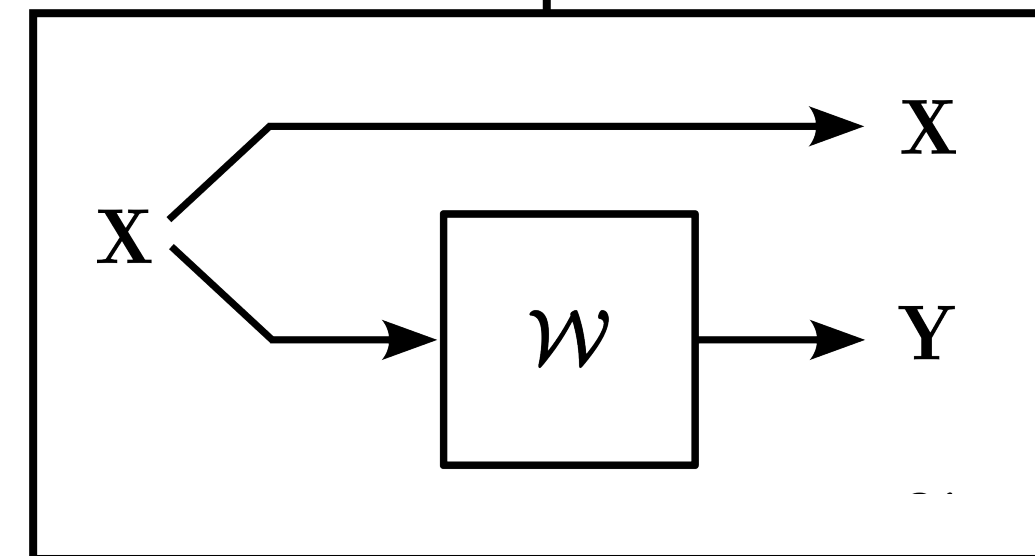
R. G. Gallager, IEEE Trans. Inform. Th. (1965)

Umlaut information

for a classical channel

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

$$E_{\text{sp}}(0^+, \mathcal{W}) = \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| P_X \mathcal{W}_{Y|X}) = \max_{P_X} U(X; Y) =: U(\mathcal{W})$$



Umlaut information

for a classical channel

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

$$E_{\text{sp}}(0^+, \mathcal{W}) = U(\mathcal{W})$$

Properties

- **Alternative form**

$$U(\mathcal{W}) = -\log \min_{P_X} \sum_{y \in \mathcal{Y}} \exp \left(\mathbb{E}_{X \sim P_X} \log \mathcal{W}(y|X) \right)$$

- **Additivity**

$$U(\mathcal{W}_1 \times \mathcal{W}_2) = U(\mathcal{W}_1) + U(\mathcal{W}_2)$$



$$L(\mathcal{W}) := \max_{P_X} D(P_X P_Y \| \mathcal{W}_{Y|X} P_X)$$

$$L(\mathcal{W}_1 \times \mathcal{W}_2) \neq L(\mathcal{W}_1) + L(\mathcal{W}_2)$$

$$T(\mathcal{W}) := \max_{P_X} \min_{Q_X} \min_{Q_Y} D(Q_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

$$T(\mathcal{W}_1 \times \mathcal{W}_2) \stackrel{?}{=} T(\mathcal{W}_1) + T(\mathcal{W}_2)$$

Umlaut information

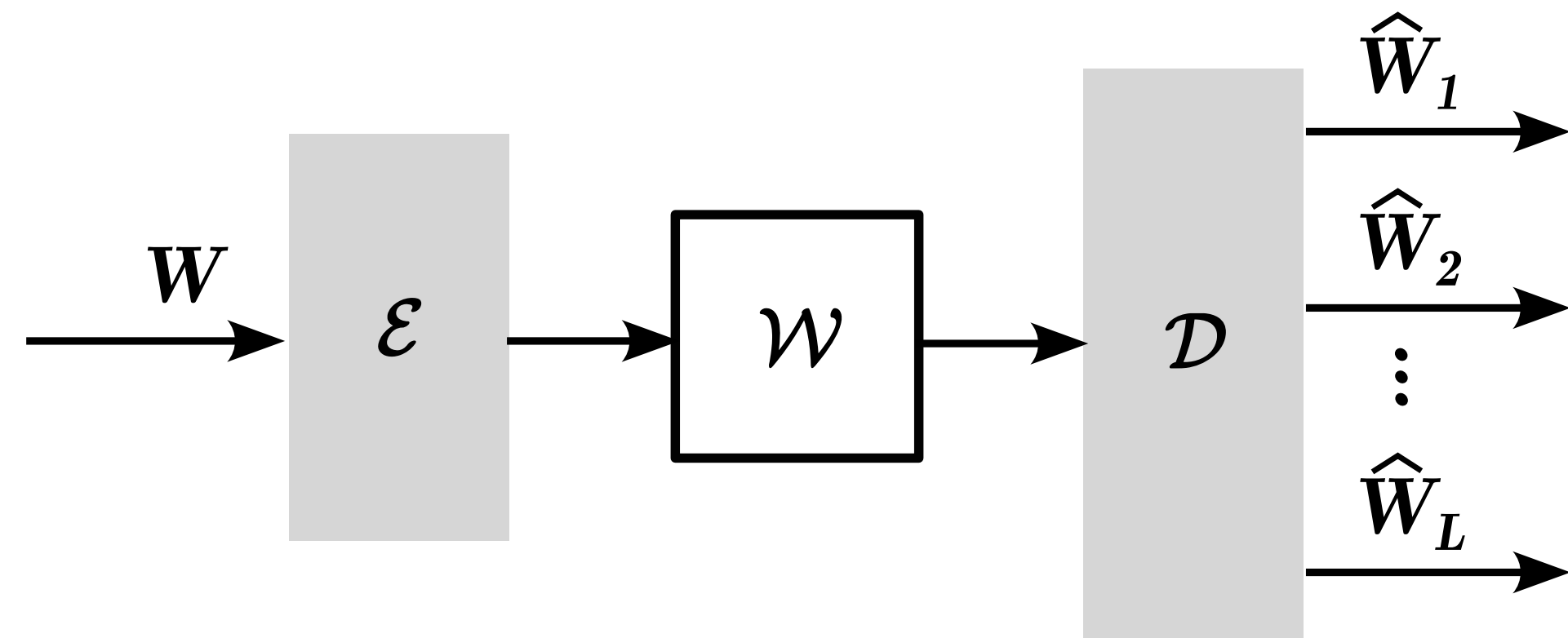
List decoding

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

$$E_{\text{sp}}(0^+, \mathcal{W}) = U(\mathcal{W})$$

$$\lim_{L \rightarrow \infty} E_L(0^+, \mathcal{W}) = U(\mathcal{W})$$

$$\left| E_L(0^+, \mathcal{W}) - U(\mathcal{W}) \right| = o\left(\sqrt{\frac{\log L}{L}}\right)$$



$$P_{\text{err}}^{(L)} = \mathbb{P}(W \notin \{\hat{W}_1, \dots, \hat{W}_L\}) \sim e^{-nE_L(r, \mathcal{W}) + o(1)}$$

Umlaut information

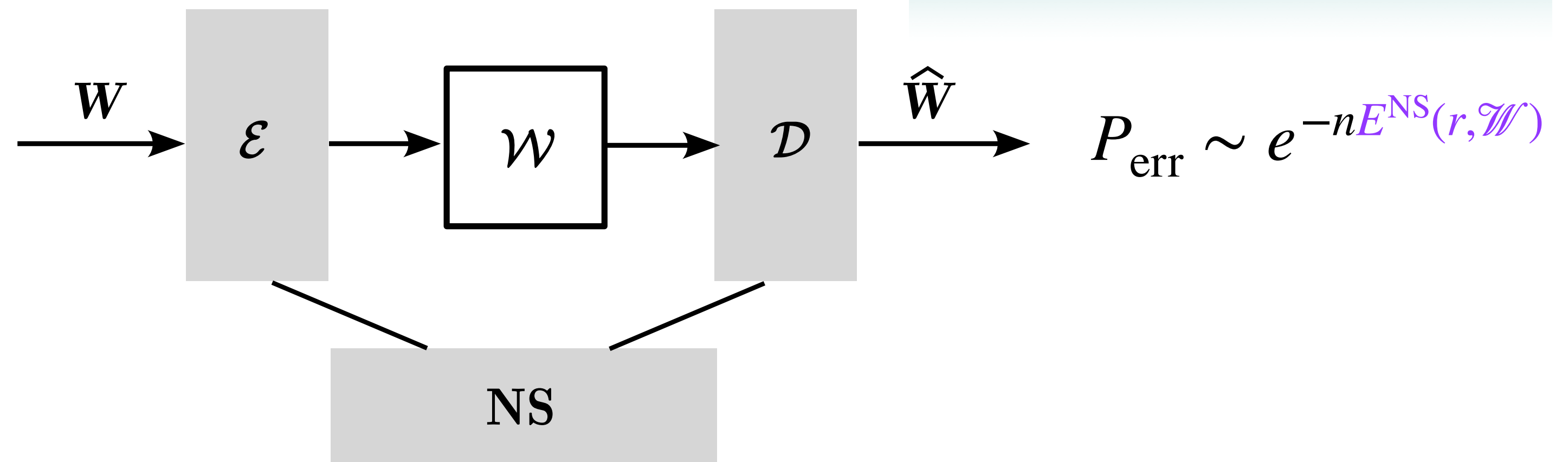
Non-signalling assisted communication

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

$$E_{\text{sp}}(0^+, \mathcal{W}) = U(\mathcal{W})$$

$$\lim_{L \rightarrow \infty} E_L(0^+, \mathcal{W}) = U(\mathcal{W})$$

$$E^{\text{NS}}(0^+, \mathcal{W}) = U(\mathcal{W})$$

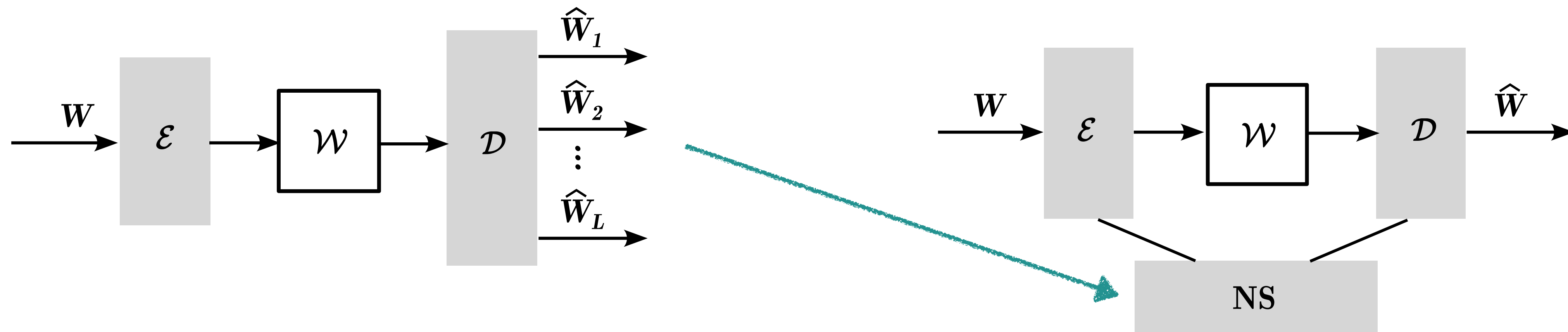


- Polyanskiy, H. V. Poor, and S. Verdú, *Channel coding rate in the finite blocklength regime*, IEEE Trans Inf. Theory 56, 2307 (2010).
- W. Matthews, *A linear program for the finite block length converse of Polyanskiy–Poor–Verdú via nonsignaling codes*, IEEE Trans. Inf. Theory 58, 7036 (2012).

Umlaut information

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

Theorem. $U(\mathcal{W}) = E_{\text{sp}}(0^+, \mathcal{W}) = E^{\text{NS}}(0^+, \mathcal{W}) = \lim_{L \rightarrow \infty} E_L(0^+, \mathcal{W})$



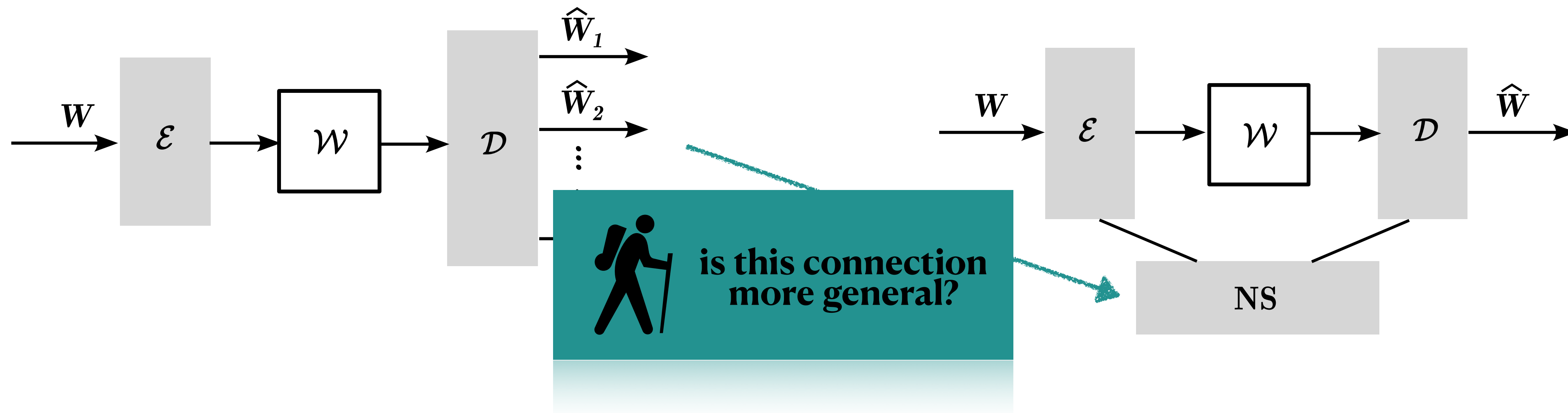
Umlaut information

$$U(\mathcal{W}) := \max_{P_X} \min_{Q_Y} D(P_X Q_Y \| \mathcal{W}_{Y|X} P_X)$$

Theorem. $U(\mathcal{W}) = E_{\text{sp}}(0^+, \mathcal{W}) = E^{\text{NS}}(0^+, \mathcal{W}) = \lim_{L \rightarrow \infty} E_L(0^+, \mathcal{W})$

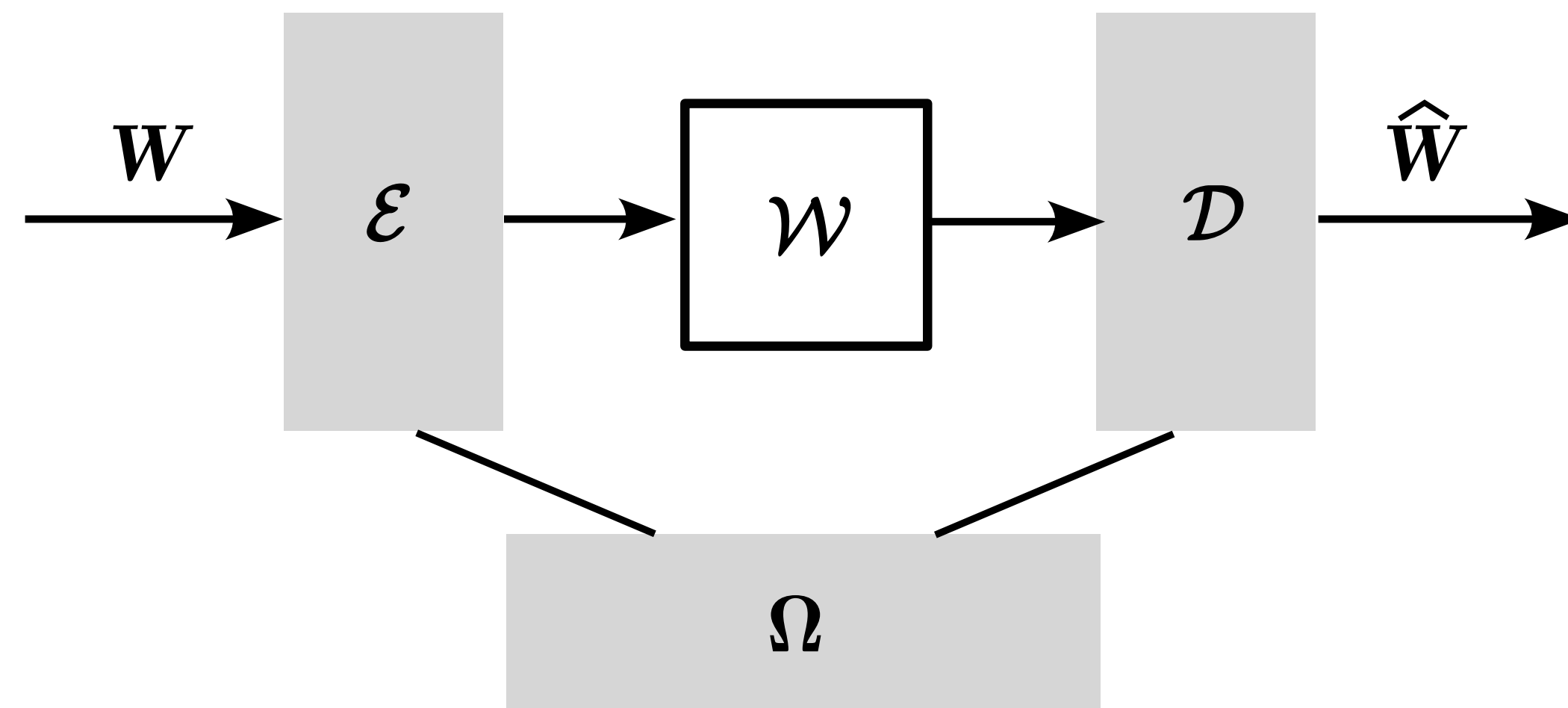
What about the
tumula information?

What about
the quantum case?



Tumula information?

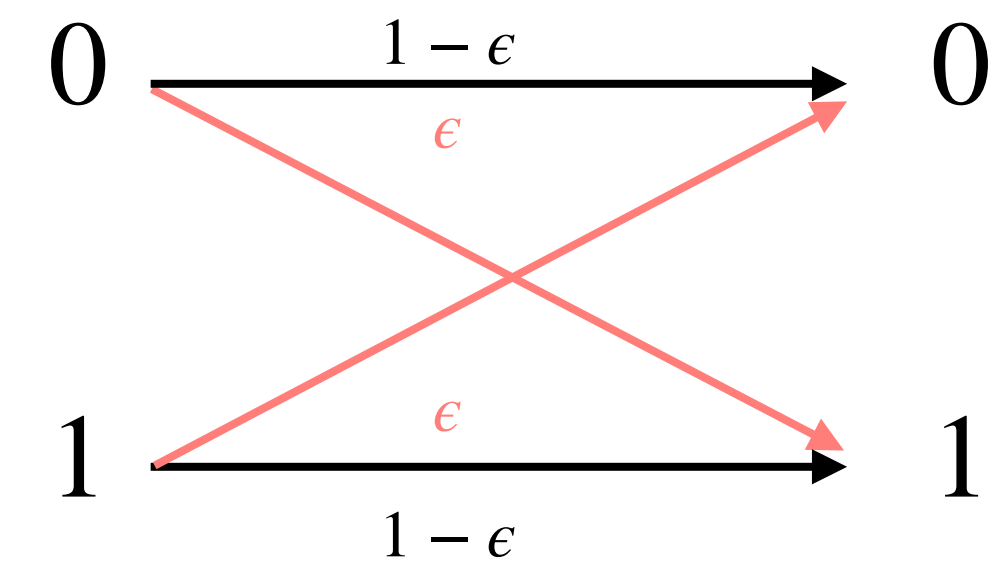
$$T(\mathcal{W}) \stackrel{?}{=} E^{\Omega}(0^+, \mathcal{W})$$



Binary symmetric channel

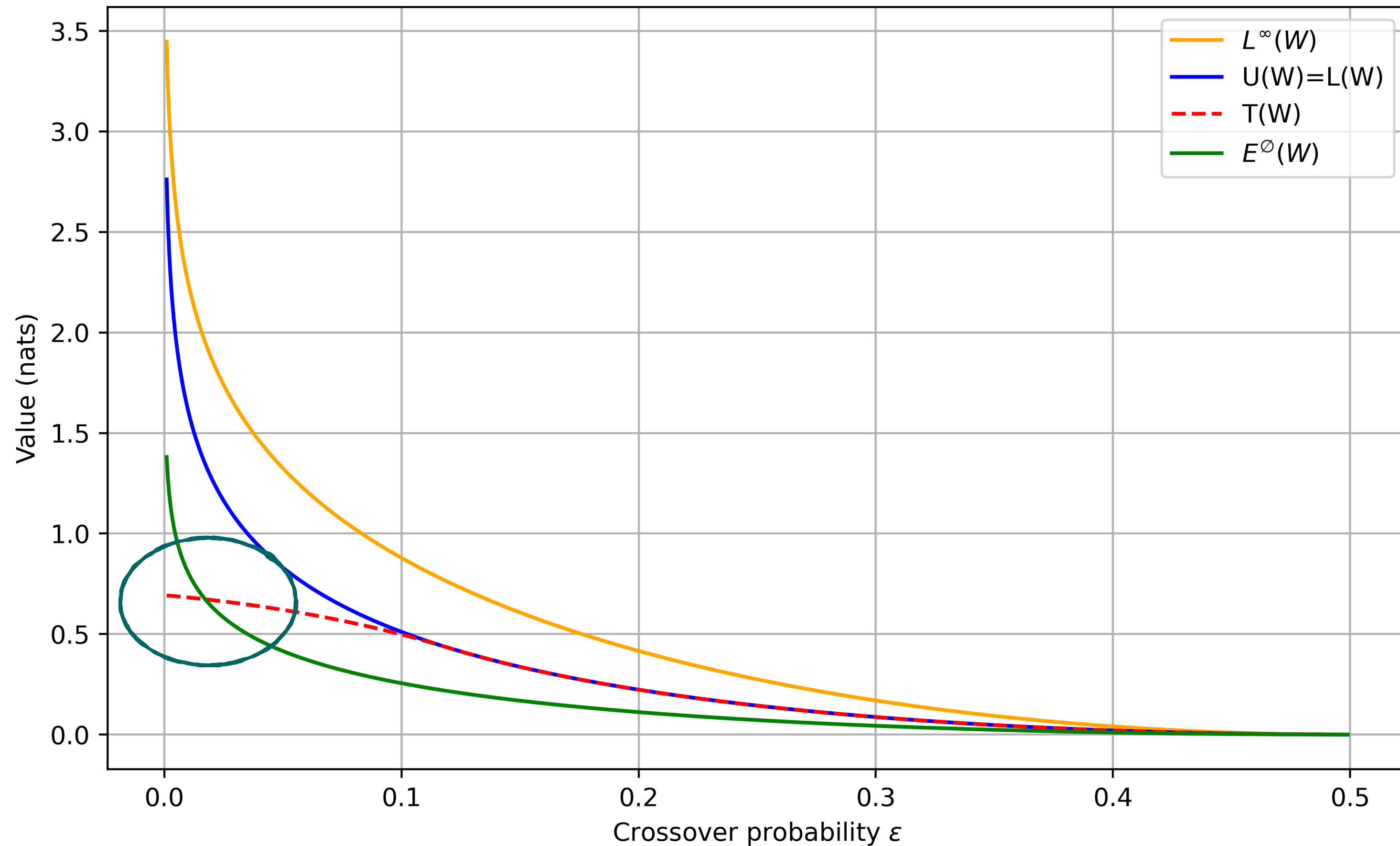
$$\mathcal{X} = \mathcal{Y} = \{0,1\}$$

$\epsilon \in [0,1]$ crossover probability



Tumula information?

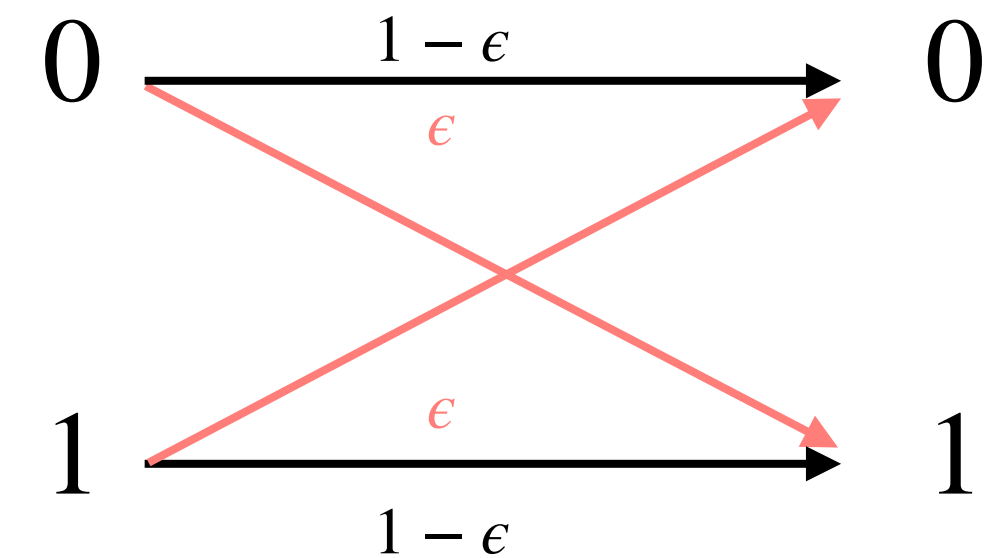
$$T(\mathcal{W}) \stackrel{?}{=} E^{\Omega}(0^+, \mathcal{W})$$



Binary symmetric channel

$$\mathcal{X} = \mathcal{Y} = \{0, 1\}$$

$\epsilon \in [0, 1]$ crossover probability



$$T(\mathcal{W}_\epsilon) < E^\emptyset(0^+, \mathcal{W}_\epsilon)$$

for sufficiently small ϵ

Quantum umlaut information

States and channels

$$U(A; B)_\rho := \min_{\sigma_B} D(\rho_A \otimes \sigma_B \| \rho_{AB})$$

quantum umlaut information
of a bipartite quantum state

$$U(\mathcal{N}) := \sup_{\Psi_{A'A}} U(A'; B)_{(\text{Id} \otimes \mathcal{N})(\Psi)}$$

quantum umlaut information
of a quantum channel

super-additivity

$$U(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq U(\mathcal{N}_1) + U(\mathcal{N}_2)$$

CQ channels

$$U(\mathcal{M}_1 \otimes \mathcal{M}_2) = U(\mathcal{M}_1) + U(\mathcal{M}_2)$$

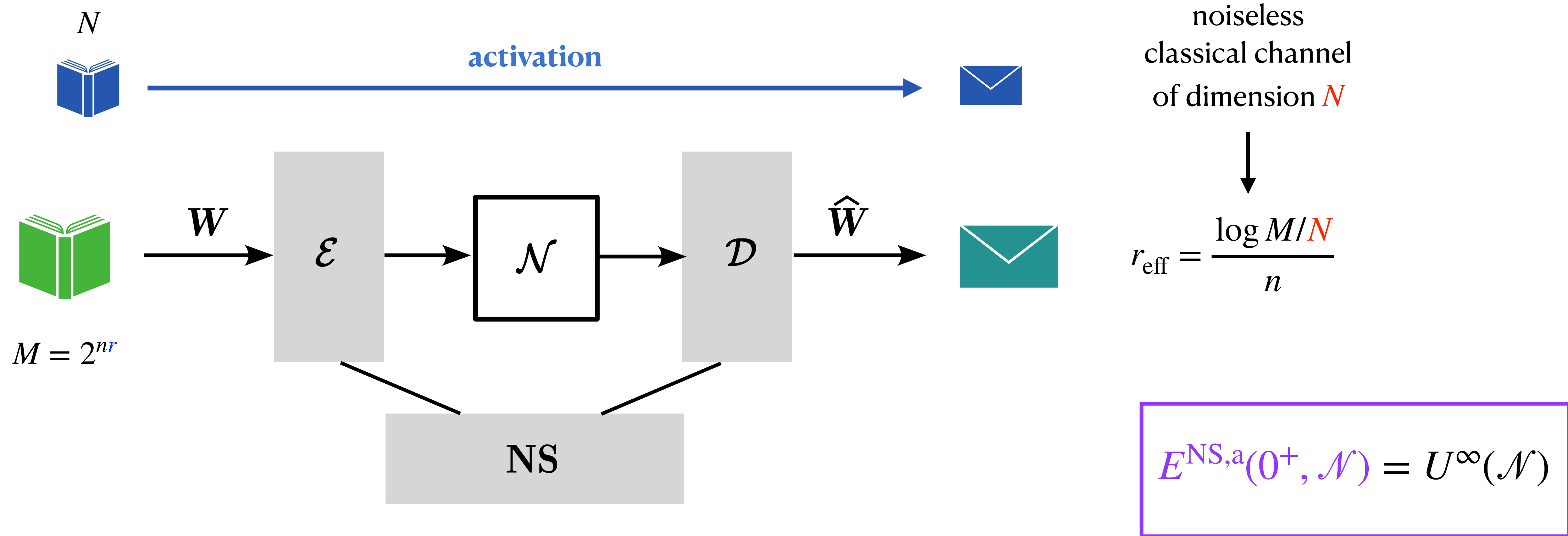
QQ channels

$$U(\mathcal{A}^{\otimes 2}) > 2U(\mathcal{A})$$

regularised channel umlaut information

$$U^\infty(\mathcal{N}) := \lim_{n \rightarrow \infty} \frac{1}{n} U(\mathcal{N}^{\otimes n})$$

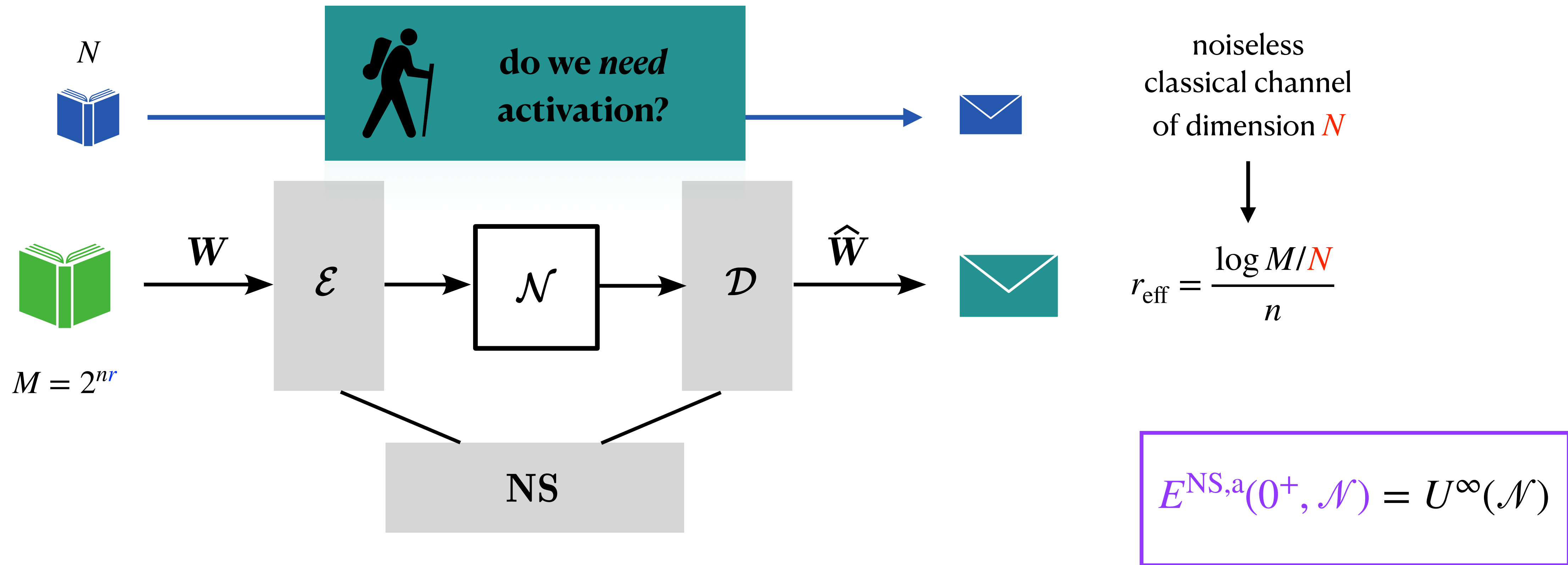
Non signalling assistance and activation



W. Matthews and S. Wehner, *Finite blocklength converse bounds for quantum channels*, IEEE Trans. Inf. Th. 60, 7317–7329 (2014)

X. Wang, K. Fang, and M. Tomamichel, *On Converse Bounds for Classical Communication Over Quantum Channels*, IEEE Trans. Inf. Th. 65, 4609 (2019)

Non signalling assistance and activation



W. Matthews and S. Wehner, *Finite blocklength converse bounds for quantum channels*, IEEE Trans. Inf. Th. 60, 7317–7329 (2014)

X. Wang, K. Fang, and M. Tomamichel, *On Converse Bounds for Classical Communication Over Quantum Channels*, IEEE Trans. Inf. Th. 65, 4609 (2019)

What about lists?

Theorem. $E_L(0^+, \mathcal{N}) \leq E^{\text{NS},a}(0^+, \mathcal{N}) = U^\infty(\mathcal{N})$



entanglement?

Main ingredient: **collapse of communication complexity via PR-boxes**
W. van Dam, *Implausible consequences of superstrong nonlocality*, Nat. Comput. (2013)

What about lists?



achievability?

Theorem. $E_L(0^+, \mathcal{N}) \leq E^{\text{NS,a}}(0^+, \mathcal{N}) = U^\infty(\mathcal{N})$



entanglement?

Classical case. $U(\mathcal{W}) = E_{\text{sp}}(0^+, \mathcal{W}) = E^{\text{NS}}(0^+, \mathcal{W}) = \lim_{L \rightarrow \infty} E_L(0^+, \mathcal{W})$

channel coding



$$P_6(\text{MUTUAL}) \supseteq \underbrace{\{\text{LAUTUM, UMLAUT, TUMULA}\}}$$

asymmetric hypothesis testing