A Reversible Quantum Resource Theory For Classical-Quantum Channels

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Quantum Resource Theories (QRTs)

Classical & Quantum Information Theories $\equiv$ examples of theories of interconversions among different resources

Resources
- quantum or classical
- static (i.e. states) or dynamic (i.e. channels)
- noiseless or noisy

e.g. Transmission of classical information through a noisy quantum channel

$\mathcal{N}$ amounts to:

(asymptotic i.i.d. limit)

(quantum, dynamic, noisy resource)

$$\mathcal{N} \xrightarrow{\text{encoding \& decoding}} \mathrm{id}_{\mathrm{classical}}$$

(classical, dynamic, noiseless resource)

$\mathcal{N}^{\otimes n}$;

$p_{\mathrm{err}}^{(n)} \to 0$ as $n \to \infty$

Optimal rate of conversion

$$r(\mathcal{N} \to \mathrm{id}_{\mathrm{classical}}) = C(\mathcal{N})$$

Classical capacity

Asymptotic Reversibility of QRTs

A key focus of a QRT: to determine the optimal rate of conversion of one resource to another under the action of free operations

Note: manipulation of
- quantum states : static QRTs
- quantum channels : dynamic QRTs

e.g. $r = r(\rho_1 \to \rho_2)$:

$\rho_1^{\otimes n} \xrightarrow{\text{free operations}} \rho_2^{\otimes \lceil rn \rceil}$ with $p_{\text{err}}^{(n)} \to 0$ as $n \to \infty$

Questions:

(i) Is the QRT asymptotically reversible? (i.e. Is $r(\rho_1 \to \rho_2) = r(\rho_2 \to \rho_1)^{-1}$?)

(ii) Is there a unique resource quantifier for the QRT ?

What are the choices of free operations under which such a reversibility holds ?

Free Operations

e.g. In the entanglement theory of pure bipartite states: free operations = LOCC

It is asymptotically reversible (distillable entanglement = entanglement cost)

More generally, for any QRT one considers

- Resource non-generating (RNG) operations:

restricted set of allowed operations that do not generate resources; free object $\not\longrightarrow$ resource

e.g. In the entanglement theory of mixed bipartite states: RNG = SEPP maps/non-entangling maps

But not asymptotically reversible under RNG operations *[Lami, Regula]*

A broader class of free operations:

- Asymptotically Resource non-generating (ARNG) operations

Asymptotically resource non-generating (ARNG) operations

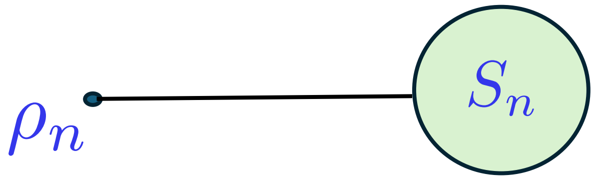
Operations that can generate some limited amount of resource when acting on a free object

but the resource generated when acting on n copies of a free object $\to 0$ as $n \to \infty$

ARNG operations: sequence of operations that are resource non-generating only in the asymptotic limit

The measure according to which the generated resource is small: log robustness

e.g. For a static QRT in which S_n : the set of free states



Log robustness of ρ_n: $\log(1+R(\rho_n)) = E_{\max}(\rho_n) := \min_{\omega_n \in S_n} D_{\max}(\rho_n || \omega_n)$

[D2009]

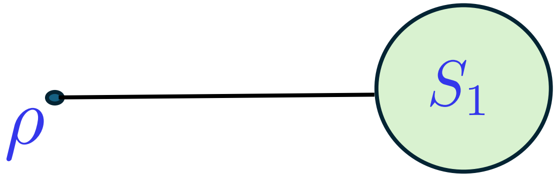
(related to the amount of another state that should be mixed with ρ_n to make it a free state)

Asymptotic Reversibility of QRTs

Asymptotic reversibility has mostly been studied for static QRTs

Resource monotones : monotonically non-increasing under free operations

e.g. For a static QRT with $(S_n)_n$: sequences of sets of free states



Relative entropy of resource: $R(\rho) := \min_{\sigma \in S_1} D(\rho||\sigma) \equiv D(\rho||S_1)$

Regularized relative entropy of resource: $R^\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} R(\rho^{\otimes n})$

Log robustness: $E_{\max}(\rho) := \min_{\sigma \in S_1} D_{\max}(\rho||\sigma)$

Theorem [Brandao & Gour 2015] : Asymptotic reversibility of static QRTs

If a static QRT has sequences of sets of free states $(S_n)_n$, that satisfy a set of 5 axioms, then

the optimal asymptotic rate of conversion is given by

$$r(\rho_1 \to \rho_2) = \frac{R^\infty(\rho_1)}{R^\infty(\rho_2)} \quad \text{.........(1)} \qquad R^\infty(\rho_i) > 0,\ i = 1, 2,$$

- Such a QRT is asymptotically reversible :

  since $r(\rho_2 \to \rho_1) \overset{(1)}{=} r(\rho_1 \to \rho_2)^{-1}$

- R^∞ : unique quantifier of resource for the QRT

Key ingredient: Generalized Quantum Stein's Lemma (GQSL)

[Brandao, Plenio]
[Hayashi, Yamasaki]
[Lami]

Asymptotic Reversibility of Dynamic QRTs

Aim of this talk: Establish asymptotic reversibility of a QRT of classical-quantum channels

Note: a concurrent and independent work was done in

(1) **Generalized Quantum Stein's Lemma for Classical-Quantum Dynamical Resources**
Masahito Hayashi, Hayata Yamasaki; quant-ph arXiv:2509.07271

(talk of Hayata Yamasaki later today)

QRT of classical-quantum (c-q) channels

A c-q channel:

$$\mathcal{E} : \mathcal{X} \to A$$
$$x \mapsto \mathcal{E}(x) \in \mathcal{D}(A)$$

$\mathcal{X}$: a finite alphabet

A : a quantum system with a finite-diml Hilbert space $\mathcal{H}_A$

$\mathcal{D}(A) \equiv \mathcal{D}(\mathcal{H}_A)$ set of quantum states of A

We can assign a finite-dimensional Hilbert space to $\mathcal{X}$:

$$\mathcal{X} \leftrightarrow \mathcal{H}_{\mathcal{X}} := \mathrm{span}\{|x\rangle \mid x \in \mathcal{X}\}$$

$\{|x\rangle\}_{x\in\mathcal{X}}$: orthonormal basis of $\mathcal{H}_X$

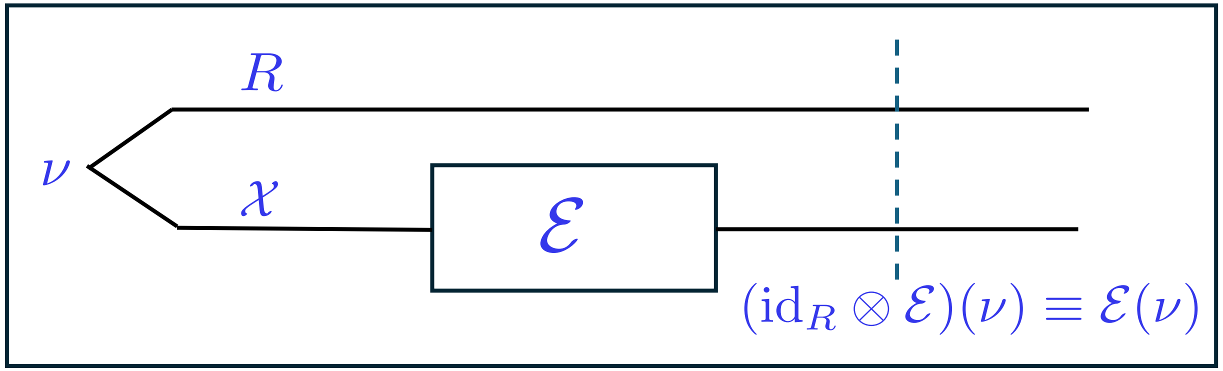
Note: A c-q channel can be viewed as a fully quantum channel in which the input quantum states are first subjected to a projective measurement in this basis $\{|x\rangle\}_{x\in\mathcal{X}}$ $\mathcal{E} : \mathcal{X} \to A$

$\mathrm{CQ}(\mathcal{X} \to A)$: **set of c-q channels**

$$\forall \rho \in \mathcal{D}(\mathcal{X}) \equiv \mathcal{D}(\mathcal{H}_{\mathcal{X}}), \;\; \mathcal{E}(\rho) = \sum_{x\in\mathcal{X}} \mathrm{Tr}[\rho\, |x\rangle\langle x|]\omega_x \in \mathrm{CQ}(\mathcal{X} \to A) : \omega_x \in \mathcal{D}(A)\; \forall x \in \mathcal{X}$$

c-q channels contd.

Let $\mathcal{D}(R\mathcal{X})$: set of quantum states on $\mathcal{H}_R \otimes \mathcal{H}_\mathcal{X}$, where R : a reference system



Note: the state $\nu \in \mathcal{D}(R\mathcal{X})$ can be entangled.

Distance between c-q channels: For $\mathcal{E}, \mathcal{F} \in \mathrm{CQ}(\mathcal{X} \to A)$:

$$\|\mathcal{E} - \mathcal{F}\|_\Diamond := \sup_{\nu \in \mathcal{D}(RX)} \|(\mathrm{id}_R \otimes \mathcal{E})(\nu) - (\mathrm{id}_R \otimes \mathcal{F})(\nu)\|_1$$

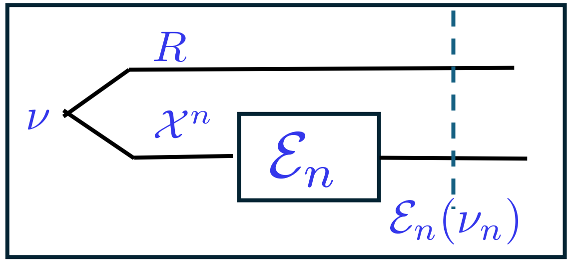
QRT of classical-quantum (c-q) channels

- **Free objects:** Free c-q channels $\mathcal{S}_n \subset \mathrm{CQ}(\mathcal{X}^n \to A^n)$: satisfying certain axioms
- **Resources :** c-q channels in $\mathrm{CQ}(\mathcal{X}^n \to A^n)$ that $\notin \mathcal{S}_n$
- **Log robustness :** $\forall\, \mathcal{E}_n \subset \mathrm{CQ}(\mathcal{X}^n \to A^n)$:

$$E_{\max}(\rho_n) := \min_{\sigma_n \in S_n} D_{\max}(\rho_n \| \sigma_n)$$

$$E_{\max}(\mathcal{E}_n) := \inf_{\mathcal{F}_n \in \mathcal{S}_n} D_{\max}(\mathcal{E}_n \| \mathcal{F}_n) = \inf_{\mathcal{F}_n \in \mathcal{S}_n} \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} D_{\max}(\mathcal{E}_n(\nu_n) \| \mathcal{F}_n(\nu_n))$$

- **Relative entropy of resource:** $\forall\, \mathcal{E}_n \subset \mathrm{CQ}(\mathcal{X}^n \to A^n)$:



$$R(\mathcal{E}_n) := D(\mathcal{E}_n \| \mathcal{S}_n) = \inf_{\mathcal{F}_n \in \mathcal{S}_n} \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} D(\mathcal{E}_n(\nu_n) \| \mathcal{F}_n(\nu_n))$$

- **Regularized relative entropy of resource:** $\forall\, \mathcal{E} \subset \mathrm{CQ}(\mathcal{X} \to A)$:

$$R^{\infty}(\mathcal{E}) := \lim_{n \to \infty} \frac{1}{n} R(\mathcal{E}^{\otimes n})$$

$$R(\rho_n) := D(\rho_n \| S_n)$$

QRT of classical-quantum (c-q) channels

Free operations: $\tilde{\mathcal{O}} = (\Theta_n)_n$ sequences of superchannels that are ARNG

Superchannels: $\Theta_n : \mathrm{CQ}(\mathcal{X}^n \to A^n) \to \mathrm{CQ}(\mathcal{X}^n \to A^n) :$

$\forall\, \mathcal{E}_n \in \mathrm{CQ}(\mathcal{X}^n \to A^n),$

$$\Theta_n(\mathcal{E}_n) := \mathcal{N}_n^{post} \circ \mathcal{E}_n \circ \mathcal{N}_n^{pre}$$

$(\Theta_n)_n$ is ARNG if for any sequence of free c-q channels $(\mathcal{F}_n \in \mathcal{S}_n)_n,$

$$\lim_{n\to\infty} D_{\max}(\Theta_n(\mathcal{F}_n) \| \mathcal{S}_n) = 0$$

QRT of classical-quantum (c-q) channels

Our Aim:

To find the optimal asymptotic conversion rate between two resources $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{CQ}(\mathcal{X} \to A)$,

under ARNG operations

i.e. the optimal r for which $\exists\, (\Theta_n)_n \in \tilde{\mathcal{O}}$ s.t.

$$\mathcal{E}_1^{\otimes n} \mapsto \Theta_n(\mathcal{E}_1^{\otimes n}) \approx \mathcal{E}_2^{\otimes \lceil rn \rceil}$$

with vanishing error in the limit $n \to \infty$

$$\left\| \Theta_n(\mathcal{E}_1^{\otimes n}) - \mathcal{E}_2^{\otimes \lceil rn \rceil} \right\|_{\diamond} \to 0 \quad \text{as} \quad n \to \infty$$

Main Result: Asymptotic Reversibility of the QRT of c-q channels

Theorem 1: If the sequences of sets of free channels $(\mathcal{S}_n \subset \mathrm{CQ}(\mathcal{X}^n \to A^n))_n$ satisfy the axioms,

& the set of free operations are ARNG,

then for any two c-q channels, $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{CQ}(\mathcal{X} \to A)$ with $R^\infty(\mathcal{E}_i) > 0, \quad i = 1, 2,$

optimal asymptotic conversion rate

$$r(\mathcal{E}_1 \to \mathcal{E}_2) = \frac{R^\infty(\mathcal{E}_1)}{R^\infty(\mathcal{E}_2)}$$

Hence the QRT is asymptotically reversible, i.e. $r(\mathcal{E}_1 \to \mathcal{E}_2) = r(\mathcal{E}_2 \to \mathcal{E}_1)^{-1}$

& $R^\infty(\mathcal{E})$ is the unique resource quantifier of the QRT

Main ingredient of the proof

- **Generalized Quantum Stein's Lemma (GQSL) for c-q channels**

[Hayashi, Yamasaki 2025]
[Bergh, D, Khaitan 2025]

A crucial lemma which was pivotal in

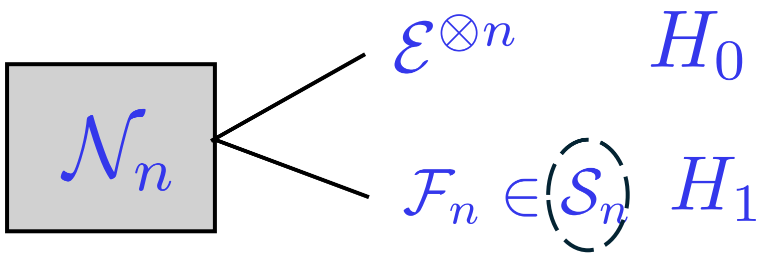
- our proof of the GQSL for c-q channels
- the proof of asymptotic reversibility (Theorem 1)

- **Lemma** [Exchange lemma]: Let $\mathcal{S}, \mathcal{T} \in \mathrm{CPTP}(A \to B)$ be 2 closed, convex sets of channels,

then
$$\inf_{\substack{\mathcal{E}\in\mathcal{S}\\ \mathcal{F}\in\mathcal{T}}} \sup_{\nu \in \mathcal{D}(R\mathcal{X})} D(\mathcal{E}(\nu)\|\mathcal{F}(\nu)) = \sup_{\nu \in \mathcal{D}(R\mathcal{X})} \inf_{\substack{\mathcal{E}\in\mathcal{S}\\ \mathcal{F}\in\mathcal{T}}} D(\mathcal{E}(\nu)\|\mathcal{F}(\nu))$$

[Gour, Winter 2019]
[Bergh, D, Salzmann 2024]

Generalized Quantum Stein's Lemma (GQSL) for c-q channels

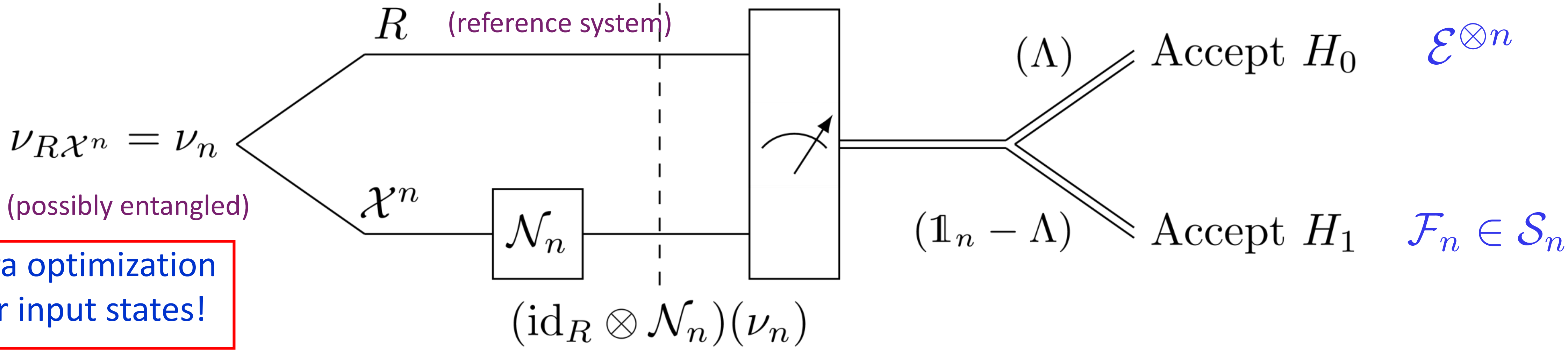


Null Hypothesis

Alternative Hypothesis (composite)

unknown c-q channel

(satisfies a certain set of axioms)



extra optimization over input states!

[since, in general, the discrimination can be improved by using such entangled states.]

Generalized Quantum Stein's Lemma (GQSL) for c-q channels

Quantity of interest:

$$\forall\, \varepsilon \in (0,1), \quad \beta_\varepsilon(\mathcal{E}^{\otimes n}, \mathcal{S}_n) := \text{ optimal } P(H_1|H_0) \text{ such that } P(H_0|H_1) \leq \varepsilon$$

(type-II error prob.) (type-I error prob.)

[Hayashi, Yamasaki] *[Bergh, D, Khaitan]*

Theorem [GQSL for CQ channels] : If the sequences of sets of c-q channels $(\mathcal{S}_n)_n$ satisfy the axioms, then:

$$\lim_{n \to \infty} -\frac{1}{n} \log \beta_\varepsilon(\mathcal{E}^{\otimes n}, \mathcal{S}_n) = \lim_{n \to \infty} \frac{1}{n} D(\mathcal{E}^{\otimes n} \| \mathcal{S}_n)$$

$$D(\mathcal{E}^{\otimes n} \| \mathcal{S}_n) := \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} \inf_{\mathcal{F}_n \in \mathcal{S}_n} D(\mathcal{E}^{\otimes n} \| \mathcal{S}_n)$$

Generalized Quantum Stein's Lemma (GQSL) for c-q channels

Quantity of interest:

$$\forall\, \varepsilon \in (0,1), \quad \beta_\varepsilon(\mathcal{E}^{\otimes n}, \mathcal{S}_n) := \text{ optimal } P(H_1|H_0) \text{ such that } P(H_0|H_1) \le \varepsilon$$

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[Hayashi, Yamasaki] *[Bergh, D, Khaitan]*

Theorem [GQSL for CQ channels] : If the sequences of sets of c-q channels $(\mathcal{S}_n)_n$ satisfy the axioms, then:

$$\lim_{n\to\infty} -\frac{1}{n}\log \beta_\varepsilon(\mathcal{E}^{\otimes n}, \mathcal{S}_n) = \lim_{n\to\infty} \frac{1}{n} D(\mathcal{E}^{\otimes n}\|\mathcal{S}_n)$$

$$\beta_\varepsilon(\mathcal{E}^{\otimes n}, \mathcal{S}_n) \sim 2^{-n\Delta}$$

How is the GQSL for c-q channels at all related to the asymptotic reversibility of our QRT?

Note:

The axioms for the composite hypothesis in the GQSL

must be the same as

The axioms for the set of free states in the QRT

Theorem 1: Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{CQ}(\mathcal{X} \to A)$ be two c-q channels, with $R^\infty(\mathcal{E}_i) > 0, \quad i = 1, 2.$

If the set of free channels satisfy the axioms, then $r(\mathcal{E}_1 \to \mathcal{E}_2) = \frac{R^\infty(\mathcal{E}_1)}{R^\infty(\mathcal{E}_2)}$ ....(I)

$$R^\infty(\mathcal{E}_1) := \lim_{n\to\infty} \frac{1}{n} R(\mathcal{E}_1^{\otimes n}) = \lim_{n\to\infty} \frac{1}{n} \inf_{\mathcal{F}_n \in \mathcal{S}_n} \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} D(\mathcal{E}_1^{\otimes n}(\nu_n) \| \mathcal{F}_n(\nu_n)) \quad \text{...(a)}$$

Theorem (GQSL) : $\forall \varepsilon \in (0,1), \lim_{n\to\infty} -\frac{1}{n} \log \beta_\varepsilon(\mathcal{E}_1^{\otimes n}, \mathcal{S}_n) = \lim_{n\to\infty} \frac{1}{n} D(\mathcal{E}^{\otimes n} \| \mathcal{S}_n)$......(II)

RHS of (II) $= \lim_{n\to\infty} \frac{1}{n} \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} \inf_{\mathcal{F}_n \in \mathcal{S}_n} D(\mathcal{E}_1^{\otimes n}(\nu_n) \| \mathcal{F}_n(\nu_n))$

[Exchange Lemma]

$= \lim_{n\to\infty} \frac{1}{n} \inf_{\mathcal{F}_n \in \mathcal{S}_n} \sup_{\nu_n \in \mathcal{D}(R\mathcal{X}^n)} D(\mathcal{E}^{\otimes n}(\nu_n) \| \mathcal{F}_n(\nu_n))$ ...(b)

$\Longrightarrow$ $R^\infty(\mathcal{E}_1) =$ **RHS of the GQSL for c-q channels!**

Examples of QRTs of c-q channels to which our result apply

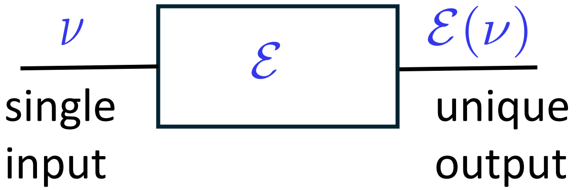
Example 1: [free c-q channels: those which yield separable output states]

$$\mathcal{S}_n := \{ \mathcal{F} \in \mathrm{CQ}(\mathcal{X}^n \to A^n B^n) \mid \mathcal{F}(|x\rangle\langle x|) \in \mathrm{SEP}(A^n : B^n) \, \forall x \in \mathcal{X}^n \}$$

Note: If $|\mathcal{X}| = 1$ then the above construction reduces to a static QRT : the **QRT of entanglement**

Reasons:

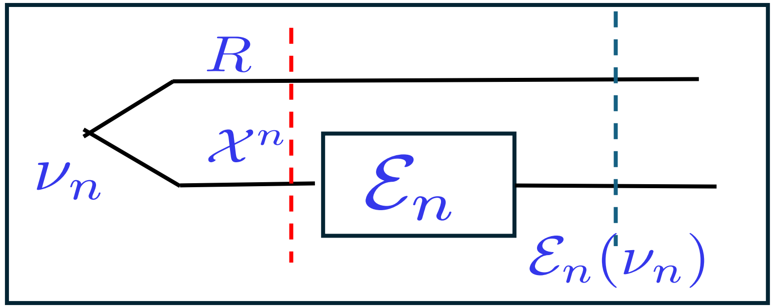
(1)



$\therefore \; \mathcal{E} \longleftrightarrow \mathcal{E}(\nu)$

channel   state

(2) the reference system R does not play a role in this case



e.g. $\mathcal{X} = \{0\} \implies \mathcal{D}(\mathcal{H}_{\mathcal{X}}) = \{|0\rangle\langle 0|\}$

$\implies$ **any input** is of the form $\nu_n = \nu_R \otimes |0\rangle\langle 0|^{\otimes n}$

$D, D_{\max}$ are faithful & additive under tensor products - thus we can ignore the reference system R

$\implies$ **Example 1** $(|\mathcal{X}| > 1)$ = extension of the QRT of entanglement to c-q channels

Examples of QRTs of c-q channels to which our result apply

Note: Further examples can be constructed from any resource theory for quantum states that satisfy the original Brandao-Plenio axioms

This includes the resource theories of asymmetry, athermality, coherence, magic,….

Example 2 : [free c-q channels = those yielding free states of other reversible static QRTs]

Let $(S_n)_n$: sequences of sets of states that satisfy the Brandao-Plenio axioms

They hence define the free states of an asymptotically reversible static QRT

Then,

$$\mathcal{S}_n := \{ \mathcal{F} \in \mathrm{CQ}(\mathcal{X}^n \to A^n B^n) \mid \mathcal{F}(|x\rangle\langle x|) \in S_n \, \forall x \in \mathcal{X}^n \}$$

serves as the set of free c-q channels for a reversible QRT of c-q channels

Examples of QRTs of c-q channels to which our result apply

Example 3 : [free c-q channels are all replacer channels] ⟹ [QRT of capacity of a c-q channel]

$$\mathcal{S}_n := \{ \mathcal{F}_n \in \mathrm{CQ}(\mathcal{X}^n \to A^n) \mid \exists \sigma_n \in \mathcal{D}(A^n) : \mathcal{F}_n(|x\rangle\langle x|) = \sigma_n \, \forall x \in \mathcal{X}^n \}$$

In this case, $R^\infty(\mathcal{E}) = R(\mathcal{E}) = C(\mathcal{E})$ = capacity of $\mathcal{E}$

Intuition: Every c-q channel with zero capacity is a replacer channel

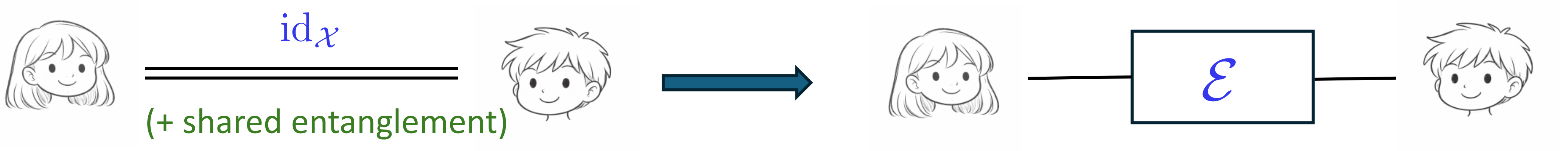
$\therefore$ $\mathcal{S}_n$: are exactly the free sets in the QRT of capacity for c-q channels

By our result : this QRT is asymptotically reversible under ARNG

This yields a Reverse Shannon Theorem for c-q channels

(Q) What does this mean and why is it true?

Reverse Shannon Theorem (for c-q channels)



(+ local pre- and post-processing)

simulate the action of $\mathcal{E}$ ....(a)

The amount of classical communication needed $= C(\mathcal{E})$

[Bennett et al]
[Berta et al]

Note: In QRT of **Example 3**: free channels = replacer channels

Local pre- & post-processing (in presence of shared entanglement) = RNG operations

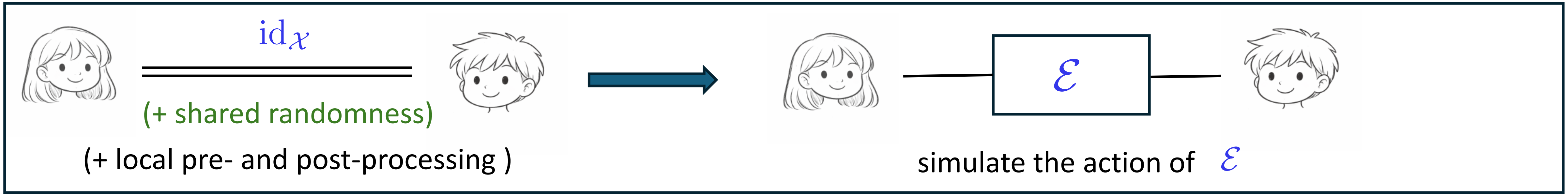
(since they leave replacer channels unaltered)

(a) : $\mathrm{id}_{\mathcal{X}} \to \mathcal{E}$ under free operations

The rate of classical communication needed for Alice and Bob to simulate the action of $\mathcal{E}$

$$= r(\mathrm{id}_{\mathcal{X}} \to \mathcal{E})$$

Reverse Shannon Theorem (for c-q channels)



What is $r(\mathrm{id}_{\mathcal{X}} \to \mathcal{E})$?

(in **Example 3**)

By our **Theorem 1:** $r(\mathcal{E}_1 \to \mathcal{E}_2) = \frac{R^\infty(\mathcal{E}_1)}{R^\infty(\mathcal{E}_2)} = \frac{C(\mathcal{E}_1)}{C(\mathcal{E}_2)}$ ……(1) since $R^\infty(\mathcal{E}) = C(\mathcal{E})$

By definition: $C(\mathcal{E}) \equiv r(\mathcal{E} \longrightarrow \mathrm{id}_{\mathcal{X}})$

Set $\mathcal{E}_1 = \mathrm{id}_{\mathcal{X}},\ \mathcal{E}_2 = \mathcal{E}$ in (1) ; $C(\mathcal{E}_1) = 1$

$$\Longrightarrow \boxed{r(\mathrm{id}_{\mathcal{X}} \to \mathcal{E}) \equiv C(\mathcal{E})^{-1}}$$

■

Summary

- Established a QRT for c-q channels & proved that it was asymptotically reversible

- **Main tool** : GQSL for c-q channel

[Hayashi, Yamasaki 2025]
[Bergh, D, Khaitan 2025]

- **Examples of our QRT :**

(1) those for which the outputs of the free channels are free states of an asymptotically reversible static Q

(2) QRT of entanglement = a special case of our QRT

(3) QRT of capacity of c-q channels ⟹ Reverse Shannon Theorem of c-q channels

Main Open Question

Can one prove a GQSL for fully quantum channels?

Thank you for your attention!

Thanks to Anirudh & Bjarne

Thanks to Bartosz & Ryuji for organizing this excellent workshop!

The Axioms for Free c-q channels

Set of **free c-q channels**:

$(\mathcal{S}_n \subset \mathrm{CQ}(\mathcal{X}^n \to A^n))_n$: sequences of sets of c-q channels, satisfying the following axioms:

- $\mathcal{S}_n$ is closed and convex as a subset of $\mathrm{CQ}(\mathcal{X}^n \to A^n)$ for each $n \in \mathbb{N}$
- $\mathcal{S}_n$ is closed under permutations of the n inputs and outputs for each $n \in \mathbb{N}$

  $\forall n \in \mathbb{N}$ & $\forall \mathcal{F}_n \in \mathcal{S}_n,$ for each permutation $\pi \in \mathfrak{S}_n$ the permuted channel $(\pi \cdot \mathcal{F}_n)$

  $\omega \mapsto (\pi \cdot \mathcal{F}_n)(\omega) = P_A(\pi)^\dagger \mathcal{F}_n(P_\mathcal{X}(\pi)\omega P_\mathcal{X}(\pi)^\dagger)P_A(\pi)$ is also an element of $\mathcal{S}_n$

- $(\mathcal{S}_n)_n$ is closed as a sequence under tensor products

  For any $m, n \in \mathbb{N},$ if $\mathcal{F}_n \in \mathcal{S}$ $, \mathcal{F}_m \in \mathcal{S}_m,$ then $\mathcal{F}_n \otimes \mathcal{F}_m \in \mathcal{S}_{n+m}$

- There exists a channel $\mathcal{F}_* \in \mathcal{S}_1$ such that the Choi state of $\mathcal{F}_*$ has full rank.

  This can be shown to be equivalent to the condition:

  $\exists c > 0,$ s.t. $D_{\max}(\mathcal{E} \| \mathcal{F}_*) \leq c, \ \forall \mathcal{E} \in \mathrm{CQ}[\mathcal{X} \to A]$