Tight any-shot quantum decoupling

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Task: Quantum decoupling

Quantum decoupling statement

- For bipartite quantum states ρ_{AE}, we are after upper bounds on *decoupling error*

$$\delta_{\mathrm{dec}}(\rho_{AE}, \mathcal{E}_{A\to C}) \coloneqq \int \mathrm{dist}\Big[\big((\mathcal{E}_{A\to C} \circ U_A) \otimes I_E\big)(\rho_{AE}), \omega_C \otimes \rho_E\Big]\mathrm{d}U$$

for decoupling maps $\mathcal{E}_{A\to C}$ with corresponding Choi state $\omega_{A'C} \coloneqq (I_{A'} \otimes \mathcal{E}_{A\to C})(\Phi_{A'A})$

- *No catalysts* and distance measure $\mathrm{dist}[\cdot]$, e.g., as

$T(\rho,\sigma) \coloneqq \frac{1}{2}\|\rho - \sigma\|_1$ trace distance $\Rightarrow \delta_{\mathrm{dec}}^T$ error

$P(\rho,\sigma) \coloneqq \sqrt{1 - \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2}$ purified distance $\Rightarrow \delta_{\mathrm{dec}}^P$ error

$D(\rho\|\sigma) \coloneqq \mathrm{Tr}[\rho(\log(\rho) - \log(\sigma))]$ relative entropy distance $\Rightarrow \delta_{\mathrm{dec}}^D$ error

- Decoupling map $\mathcal{E}_{A\to C}(\cdot)$, e.g., as partial trace $\mathrm{Tr}_{A_2}[\cdot]$ for $A = A_1A_2$ or projector $P_{A\to A_1}$

Applications of quantum decoupling

- **Quantum information theory**

  Basic technical primitive $\cong$ fully quantum version of randomness extraction against quantum side information (aka privacy amplification)

- **Quantum Shannon theory**

  Protocols for *fully quantum tasks*, e.g., quantum versions of data compression, Slepian-Wolf coding, noisy channel coding, cryptography, network tasks, etc.

- **Quantum resource theories**

  Protocols for distillation in entanglement theory, coherence theory, etc.

- **Quantum physics**

  Correlations in & efficient description of many body systems, thermalization, scrambling + black holes, etc.

What is known?

Example decoupling theorem

- Fact: Trace distance $T(\cdot)$ version [Dupuis *et al.* (**B.**) CMP 14] [Horodecki *et al.* Nature 05]

$$\delta_{\mathrm{dec}}^{T}(\rho_{AE}, \mathcal{E}_{A\to C}) \leq 2^{-\frac{1}{2}\left(\widetilde{H}_2^{\uparrow}(A|E)_\rho + \widetilde{H}_2^{\uparrow}(A'|C)_\omega\right)}$$

  with $\widetilde{H}_2^{\uparrow}$ the *optimized version* of the sandwiched conditional entropy of order two

- Replace $\widetilde{H}_2^{\uparrow}$ on the rhs with smooth conditional min-entropies $H_{\min}^{\epsilon}$ for $\epsilon \geq 0$

  → at the price of additive $\mathrm{O}(\epsilon)$ error terms

- For iid resources $\rho_{AE}^{\otimes n}$ and $\mathcal{E}_{\mathrm{A}\to C}^{\otimes n}$ with $n \in \mathbb{N}$, asymptotic decoupling $\delta_{\mathrm{dec}}^{T} \to 0$ *if*

$$H(A|E)_\rho + H(A'|C)_\omega > 0 \ (*)$$

- Ex: For $A = A_1A_2$ partial trace $\mathrm{Tr}_{A_2}[\cdot]$ gives decoupling condition $H(A|E)_\rho > \log\frac{|A_1|}{|A_2|}$

- Fact: Asymptotically $\delta_{\mathrm{dec}}^{T} \to 0$ *if and only if* $(*)$ holds [Majenz *et al.* (**B.**) PRL 17]

Critical discussion

$$\delta^T_{\mathrm{dec}}(\rho_{AE}, \mathcal{E}_{A\to C}) \leq 2^{-\frac{1}{2}\left(\widetilde{H}_2^{\uparrow}(A|E)_\rho + \widetilde{H}_2^{\uparrow}(A'|C)_\omega\right)}$$

- Trace distance $T(\cdot)$ does not cover purified distance $P(\cdot)$, but applications typically need Uhlmann's theorem ☹

  → Extension to relative entropy distance $D(\cdot) \geq P(\cdot)$ with ideas from [He, Atif, Pradhan ISIT 24]

- The ω-term is often just a dimension, e.g., the map $\mathrm{Tr}_{A_2}[\cdot]$ gives $2^{-\frac{1}{2}\widetilde{H}_2^{\uparrow}(A'|C)_\omega} = \sqrt{\frac{|A_1|}{|A_2|}}$

  → Question about tight state dependence?

- Smooth conditional min-entropy $H^{\epsilon}_{\min}$ version gives $\mathrm{O}(\epsilon)$ error terms, which can be dominating for one-shot & any-shot settings ☹

  → Question about tightness beyond first-order asymptotic decoupling $\delta^T_{\mathrm{dec}} \to 0$?

(applications in quantum Shannon & resource theories)

Main result: Any-shot decoupling

Any-shot quantum decoupling

- <u>Thm</u>: Relative entropy $D(\cdot)$ distance version

$$\delta_{\text{dec}}^{D}(\rho_{AE}, \mathcal{E}_{A\to C}) \leq \inf_{0<s\leq 1} c(s) \cdot 2^{-s\left(\widetilde{H}_{1+s}(A|E)_\rho + \widetilde{H}_{1+s}(A'|C)_\omega\right)}$$

with $\widetilde{H}_{1+s}$ the *plain version* of the sandwiched conditional entropy of order $1+s$

and the function $c(s) := \frac{s^s(1-s)^{1-s}}{s}$

1. Tighter any-shot decoupling bounds *without additive fudge terms*:

   General $\widetilde{H}_{1+s}$ vs previous $\widetilde{H_2^{\uparrow}}$ allows to get $n \to \infty$ decoupling without smoothing for

$$H(A|E)_\rho + H(A'|C)_\omega > 0$$

2. For the first time *matching error exponent converse bounds*:

   Partial trace decoupling in relative entropy distance + applications in quantum Shannon & resource theories in purified distance!

Technical comments (for experts)

$$\delta^D_{\text{dec}}(\rho_{AE}, \mathcal{E}_{A\to C}) \le \inf_{0<s\le 1} c(s) \cdot 2^{-s\left(\widetilde{H}_{1+s}(A|E)_\rho + \widetilde{H}_{1+s}(A'|C)_\omega\right)}$$

our result vs previous [Cheng, Dupuis, Gao arXiv 24] [Pau Colomer & Winter CMP 24]

$$\delta^T_{\text{dec}}(\rho_{AE}, \mathcal{E}_{A\to C}) \le \inf_{0<s\le 1} 2^{-\frac{s}{1+s}\left(\widetilde{H^{\uparrow}_{1+s}}(A|E)_\rho + \widetilde{H^{\uparrow}_{1+s}}(A'|C)_\omega\right)}$$

with $\widetilde{H}_{1+s}(A|E)_\rho := -\frac{1}{s}\log \mathrm{Tr}\left[\left(\rho_B^{-s/2(1+s)}\rho_{AB}\rho_B^{-s/2(1+s)}\right)^{1+s}\right]$ vs $\widetilde{H^{\uparrow}_{1+s}}(A|E)_\rho := \sup_{\sigma_B} -\frac{1}{s}\log \mathrm{Tr}\left[\left(\sigma_B^{-s/2(1+s)}\rho_{AB}\sigma_B^{-s/2(1+s)}\right)^{1+s}\right]$

- Ordering with pre-factor $s \cdot \widetilde{H}_{1+s}(A|E)_\rho \ge \frac{s}{1+s}\widetilde{H^{\uparrow}_{1+s}}$ [Tomamichel *et al.* (**B.**) JMP 14]
- NEWS: Improvement of δ^T_{dec} to $\mathrm{H}^{\uparrow,\mathbb{M}}_{1+s}(A|E)_\rho \ge \widetilde{H^{\uparrow}_{1+s}}(A|E)_\rho$ [Regula & Tomamichel arXiv 26]
- Question: In what sense are any of these results tight?

On tightness

Towards decoupling converse

$$\delta_{\mathrm{dec}}^{D}(\rho_{AE},\mathcal{E}_{A\to C}) \leq \inf_{0<s\leq 1} c(s)\cdot 2^{-s\left(\widetilde{H}_{1+s}(A|E)_{\rho}+\widetilde{H}_{1+s}(A'|C)_{\omega}\right)}$$

- For $A = A_1A_2$ standard quantum information decoupling via partial trace $\mathrm{Tr}_{A_2}[\cdot]$ has

$$\widetilde{H}_{1+s}(A'|C)_{\omega} = 2\cdot\log|A_2| - \log|A|$$

- Cor: Asymptotic iid expansion with rate $\mathrm{r}_n := \frac{1}{n}\log|A_2^n|$ gives *decoupling error exponent*

$$\lim_{n\to\infty} -\frac{1}{n}\log\delta_{\mathrm{dec}}^{D}\left(\rho_{AE}^{\otimes n},\mathrm{Tr}_{A_2^n}\right) \geq \sup_{0<s\leq 1} s\cdot\left\{2r - \log|A| + \widetilde{H}_{1+s}(A|E)_{\rho}\right\} \; (1)$$

- Thm: We also have the converse

$$\lim_{n\to\infty} -\frac{1}{n}\log\delta_{\mathrm{dec}}^{D}\left(\rho_{AE}^{\otimes n},\mathrm{Tr}_{A_2^n}\right) \leq \sup_{s>0} s\cdot\left\{2r - \log|A| + \widetilde{H}_{1+s}(A|E)_{\rho}\right\} \; (2)$$

→ (1) ≡ (2) below critical rate! That is, for $\mathrm{r} \leq R_{\mathrm{crit}} := \frac{\mathrm{d}}{\mathrm{d}s} - \frac{s}{2}\widetilde{H}_{1+s}(A|E)_{\rho}\Big|_{s=1}$

About the proofs

Conceptually novel proof strategy

- NO trace distance $T(\cdot)$ via two-norm $\|\cdot\|_2$ and exact Haar integral [Dupuis *et al.* (**B.**) CMP 14]
- NO relative entropy distance $D(\cdot) \leq D_2 \leq \cdots$ and exact Haar integral [He, Atif, Pradhan ISIT 24]
- NO usual type complex interpolation theory [Dupuis IEEE 23] [Cheng, Dupuis, Gao arXiv 24]
- NO catalytic decoupling [Anshu *et al.* PRL 17] [Majenz *et al.* (**B.**) PRL 17] [Li & Yao CMP 24] [Cheng *et al.* arXiv 25]

- Achievability via *operator layer cake* representation (of the derivative of the logarithm)

[Cheng & Liu arXiv 25] [Cheng *et al.* arXiv 25] [QIP 26 long plenary] → with core relevant trace inequality:

$$\mathrm{Tr}\big[\rho\big(\log(\rho+\sigma)-\log(\sigma)\big)\big] \leq \cdots \leq \frac{c(s)}{s}\cdot 2^{\widetilde{D}_{1+s}(\rho\|\sigma)} \quad \forall s \in [0,1]$$

for $c(s) := \frac{s^s(1-s)^{1-s}}{s}$ and $\widetilde{D}_{1+s}(\rho\|\sigma) := \frac{1}{s}\log \mathrm{Tr}\left[\left(\sigma^{-s/2(1+s)}\rho\sigma^{-s/2(1+s)}\right)^{1+s}\right]$

- Converse specific to standard decoupling via partial trace + uses tools from [Li *et al.* IEEE 23]

Application: Quantum state merging

Task: Quantum state merging (QSM)

- Question: With free classical communication, quantify entanglement cost e of merging A from Alice to Bob?

  [Horodecki *et al.* Nature 05]

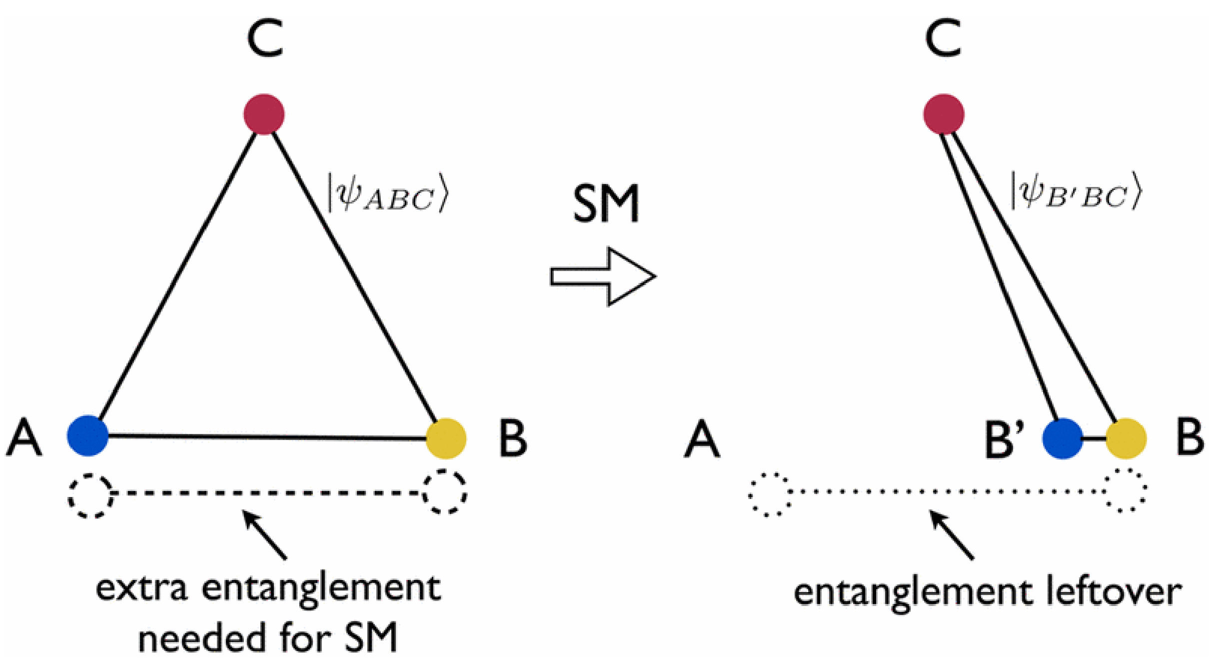
  → optimal asymptotic rate

  $$e_\infty(\psi_{ABC}) = H(A|B)_\psi = -H(A|C)_\psi$$

  → optimal one-shot cost

  $$e_1(\psi_{ABC}) \cong -H_{\min}^{\epsilon}(A|\dot{C})_\psi + \log\frac{1}{\epsilon}$$

  [**B.** Dipl. ETH 08] [Anshu *et al.* (**B.**) IEEE 20]



(picture taken from [Modi *et al.* Rev. Mod. Phys. 12])

- All basically via previous [Dupuis *et al.* (**B.**) CMP 14] → but what about error exponent of QSM?

Result QSM

- Minimal error $\delta^P(\psi_{ABC}^{\otimes n}, e)$ for QSM in purified distance $P(\cdot)$ with fixed entanglement cost e
- Thm: Error exponent of SM for (positive) entanglement cost e below critical rate E_{crit} as

$$\lim_{n\to\infty} -\frac{1}{n}\log \delta^P(\psi_{ABC}^{\otimes n}, e) = \sup_{0.5\leq\alpha<1} \frac{1-\alpha}{2\alpha}\left(e - H_\alpha^{\uparrow}(A|B)_\psi\right)$$

  with $H_\alpha^{\uparrow}$ the optimized version of the *Petz conditional entropy* of order α

- Achievability: From our new decoupling theorem applied to fixed-rank projectors as the decoupling map + used in purified distance $P(\cdot) \leq D(\cdot)$, together with Uhlmann's theorem!
- Converse: Combine arguments from [**B.** Dipl. ETH 08] [Anshu *et al.* (**B.**) IEEE 20] [Li *et al.* IEEE 22]
- Extension to other fully quantum tasks, e.g., entanglement distillation, quantum coding, etc.

(only additive single-letter converses for special cases)

Outlook

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- Main result: *Tight* quantum decoupling theorem via novel proof strategy [QIP 26 long plenary]
- Applications in quantum resource theories

  → explore more applications – exactly where *tight* decoupling matters!
- Open questions: Converse for decoupling via general maps? Beyond ensemble tight?
- NEWS: How do measured bounds $\mathrm{H}_{1+s}^{\uparrow,\mathbb{M}}(A|E)_\rho$ for $\delta_{\mathrm{dec}}^{T}$ fit in? [Regula & Tomamichel arXiv 26]

- *Hiring: PhD position at RWTH Aachen University, co-supervised with Robert Salzmann*

  → *Get in contact at robert.als.salzmann@gmail.com*

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