A resource theory of gambling

work with: Renato Renner, Jonathan Oppenheim

Maite Arcos

arXiv:2510.08418

March 20, 2026

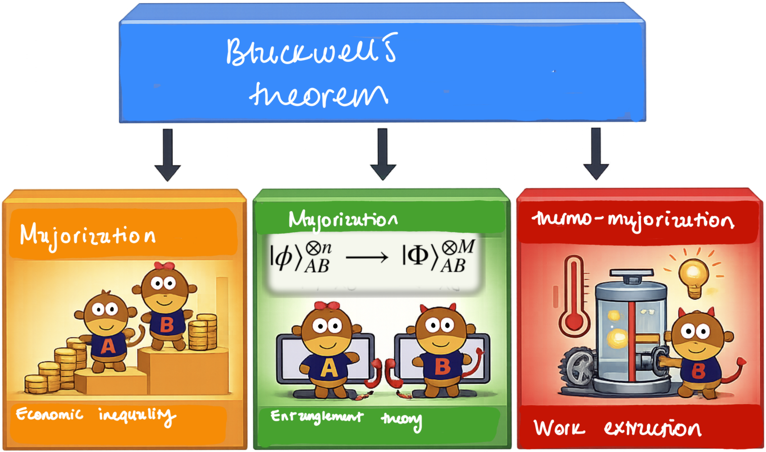
Part I

Motivation and context

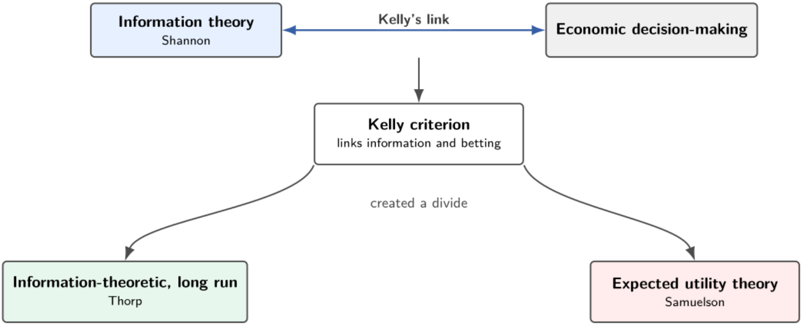
Based on: M. Arcos, R. Renner, J. Oppenheim, A resource theory of gambling
arxiv: 2510.08418

Motivation: unify perspectives

Blackwell's theorem (economics) gives a **general order on informativeness**: one experiment is better than another iff it we can simulate it by post-processing or equivalently iff it is better for a decision maker.



Context: divide between economic and information-theoretic perspectives



Main Results

1. **Single-shot Kelly gambling**
   We extend the Kelly criterion beyond the asymptotic regime to finite and one-shot betting games.
2. **Optimal strategy and success probability**
   For a target rate of return after a finite number of bets, we compute the strategy maximizing the probability of success.
3. **Risk–reward**
   Rényi divergences.
4. **Conceptual bridge**
   - expected utility (economics)
   - Information theory
5. **Connection to communication theory.**
6. **Connection to thermodynamics (with Takahiro Sagawa, Philippe Faist).**

Background

Kelly betting

Kelly betting

Kelly betting

Example: fixed odds, compare Alice's strategies, two horse race

Suppose Alice has \$100. Bob's odds (implied probs):

$$Q_X^B = (q_0^B, q_0^B) = (0.6, 0.4),$$

$$W_i = 100 \frac{q_i^A}{q_i^B}$$

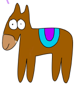
Strategy	**Bet fractions** (q_0^A, q_1^A)	**Payoff if** 1 **wins**	**Payoff if** 2 **wins**
A	$(0.9, 0.1)$	\$150	\$25
B	$(0.7, 0.3)$	\$116.67	\$75

Kelly betting

- The growth in wealth after in the asymptotic, i.i.d. limit (with re-investing) was shown by Kelly to be:

$$\frac{W_F}{W_i} = \exp\left(n\left[D(P_X \| Q_X^B) - D(P_X \| Q_X^A)\right]\right)$$

Optimal asymptotic strategy: $Q_A = P$. Ignore the odds!



Kelly betting

A response to Professor Paul A. Samuelson's objections to Kelly capital growth investing *

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November 6, 2014

Abstract

The Kelly Capital Growth Investment Strategy maximizes the expected utility of final wealth with a Bernoulli logarithmic utility function. In 1956 Kelly showed that static expected log maximization yields the maximum asymptotic long run growth. Good properties include minimizing the time to large asymptotic goals, maximizing the median, and being ahead on average after the first period. Bad properties include extremely large bets for short term favorable investment situations because the Arrow-Pratt risk aversion index is essentially zero. Paul Samuelson was a critic of this approach and I discuss his various points sent in letters he sent me and papers reprinted in MacLean, Thorp and Ziemba (2011). Samuelson's criticism is partially responsible for the current situation that most finance academics and professionals do not recommend Kelly strategies. I was asked to explain this to Fidelity Investments, a major

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Proc. Nat. Acad. Sci. USA
Vol. 68, No. 10, pp. 2493–2496, October 1971

The "Fallacy" of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling

(maximum geometric mean strategy/uniform strategies/asymptotically sufficient parameters)

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Contributed by Paul A. Samuelson, July 26, 1971

ABSTRACT Because the outcomes of repeated investments or gambles involve products of variables, authorities have repeatedly been tempted to the belief that, in a long sequence, maximization of the expected value of terminal utility can be achieved or well-approximated by a strategy of maximizing at each stage the geometric mean of outcome (or its equivalent, the expected value of the logarithm of principal plus return). The law of large numbers or of the central limit theorem as applied to the logs can validate the conclusion that a maximum-geometric-mean strategy does indeed make it "virtually certain" that, in a "long" sequence, one will end with a higher terminal wealth and utility. However, this does not imply the false corollary that the geometric-mean strategy is optimal for any finite number of periods, however long, or that it becomes asymptotically a good approximation. As a trivial counter-example, it is shown that for utility proportional to x^γ/γ, whenever $\gamma \neq 0$, the geometric strategy is suboptimal for all T and never a good approximation. For utility bounded above, as when $\gamma < 0$, the same conclusion holds. If utility is bounded above and finite at zero wealth, *no* uniform strategy can be optimal, even though it can be that the best uniform strategy will be that of the maximum geometric mean. However, asymptotically the same level of utility can be reached by an infinity of nearby uniform strategies. The true opti-

$$X_{t+1} = X_t\{w_1Y_1 + \ldots + w_nY_n\}, \sum_1^n w_j = 1$$

where the w's are the proportions decided upon for investn
in the different securities (or gambling games). The returns
dollar invested in each alternative, respectively, are sub
to known probability distributions.

$$\text{prob}\{Y_1 \leq y_1, \ldots, Y_n \leq y_n\} = F(y_1, \ldots, y_n),$$

where for simplicity the vector outcomes at any t are assu
each to be independent of outcomes at any other time peri
and to remain the same distribution over all time peri
Usually Y_t is restricted to being nonnegative to avoid ba
ruptcy. A complete decision involves selecting over the in
val of time $t = 1,2,\ldots,T$, all the vectors $[w_j(t)]$, which
be nonnegative if short-selling and bankruptcy are ru
out.

In particular, if the same strategy is followed at all tin
so that $w_j(t) \equiv w_j$, the variables become

$$X_1 = X_0x_1, X_2 = X_1x_2 = X_0x_1x_2$$

WHY WE SHOULD NOT MAKE MEAN LOG OF WEALTH BIG THOUGH YEARS TO ACT ARE LONG

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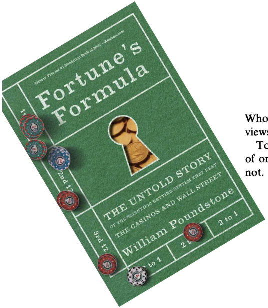
He who acts in N plays to make his mean log of wealth as big
as it can be made will, with odds that go to one as N soars,
beat me who acts to meet my own tastes for risk.

Who doubts *that*? What we do doubt[1] is that *it* should make us change our views on gains and losses – should taint our tastes for risk.

To be clear is to be found out. Know that life is not a game with net stake of one when you beat your twin, and with net stake of nought when you do not. A win of ten is not the same as a win of two. Nor is a loss of two the



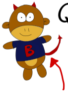
Part II

Finite n Kelly betting

Why only the type matters (already for finite n)

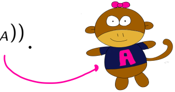
Step 1: repeated Kelly betting is a single bet on the whole sequence
Step 2: the i.i.d. source probability also depends only on the type
Hence stake, odds, and probability are all constant on each type class Λ_λ:

$$Q_A(x^n) = 2^{-n(H(\lambda_{x^n})+D(\lambda_{x^n}\|Q_A))}, \quad Q_B(x^n) = 2^{-n(H(\lambda_{x^n})+D(\lambda_{x^n}\|Q_B))},$$



and therefore

$$\frac{W_F}{W_i} = 2^{\,n(D(\lambda_{x^n}\|Q_B)-D(\lambda_{x^n}\|Q_A))}.$$



Finite-n Kelly reduces to betting on type classes.

For finite n, wealth depends on the type

$$\frac{W_F}{W_i} = 2^{n\left(D(\lambda_{x^n} \| Q_B) - D(\lambda_{x^n} \| Q_A)\right)}.$$

Each type λ defines:

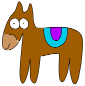
- a **reward** $D(\lambda_{x^n} \| Q_B)$
- a **probability** $2^{-nD(\lambda_{x^n} \| P)}$

Alice faces a tradeoff between reward and probability.

Optimisation Perspective

Finite–n problem:

Optimise $D(\lambda_{x^n} \| P)$ subject to a bound on $D(\lambda_{x^n} \| Q_X^B)$).



Optimal strategy:

$$Q_X^{A*}(x) \;=\; \frac{P_X(x)^{\eta}\, Q_X^B(x)^{\,1-\eta}}{\sum_{x'} P_X(x')^{\eta}\, Q_X^B(x')^{\,1-\eta}}.$$

for some $\eta \in \mathbb{R}$ which depends on the constraint.

Optimal betting strategy (Bleuer–Lapidoth–Pfister 2020):

$$Q_{A,\beta}(x) = \frac{P(x)^{1/(1-\beta)}\, Q_B(x)^{\beta/(1-\beta)}}{\sum_{x'} P(x')^{1/(1-\beta)} Q_B(x')^{\beta/(1-\beta)}}$$

Therefore:

Information-theoretic optimisation = expected utility maximisation.

Some Renyi divergences

Using the optimal strategy, we show that the wealth for a rational individual may be written as

$$\boxed{\log\left(\frac{W_F}{W_i}\right) = \alpha D(P_X || Q_X^B) + (1-\alpha) D_\alpha(P_X || Q_X^B)} \qquad (1)$$

$$\text{where} \quad \alpha = \frac{1}{1-\beta} \qquad (2)$$

Part III

A resource theory of gambling

Re-interpretation as communication task

Betting as variable-length coding

The communication interpretation of the gambling game:

- **Bob's odds** = commitment to a variable-length code: $\ell_B(z^n|y^n)$ bits from Charlie to Alice per outcome
- **Alice's bet** = her own compression scheme: $\ell_A(z^n|x^n)$ bits
- Both must satisfy Kraft equality: $\sum 2^{-\ell} = 1$ (prefix-free = unambiguous decoding)
- **Payoff = surplus communication:** $k = \ell_B - \ell_A$ bits
- Identifying $Q_A := 2^{-\ell_A}$, $Q_B := 2^{-\ell_B}$ recovers the Kelly result Q_A/Q_B

$$\frac{W_F}{W_i} = 2^{n(H(C|B) - H(C|A))}$$

Conclusions

- Information is a resource convertible into payoffs.
- **Single-shot Kelly gambling**
- **Conceptual bridge**
  - expected utility (economics)
  - Information theory
- **Connection to communication theory.**
- **Connection to thermodynamics (with Takahiro Sagawa, Philippe Faist).**
- In this setting, the payoff admits a *coding / communication interpretation*.
- Connection to thermodynamics in companion paper.
- Quantum generalisation?

Related work

Multivariate Rényi divergences characterise betting games with multiple lotteries

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(Dated: January 27, 2026)

We provide an operational interpretation of the multivariate Rényi divergence in terms of economic-theoretic tasks based on betting, risk aversion, and multiple lotteries. Specifically, we show that the multivariate Rényi divergence $D_{\underline{\alpha}}(\vec{P}_X)$ of probability distributions $\vec{P}_X = (p_X^{(0)}, \ldots, p_X^{(d)})$ and real-valued orders $\underline{\alpha} = (\alpha_0, \ldots, \alpha_d)$ quantifies the economic-theoretic value that a rational agent assigns to d lotteries with odds $o_X^{(k)} \propto (p_X^{(k)})^{-1}$ $(k = 1, \ldots, d)$ on a random event described by $p_X^{(0)}$. In particular, when the odds are fair and the rational agent is allowed to maximise over all betting strategies, the economic-theoretic value (precisely the isoelastic certainty equivalent) that the agent assigns to the lotteries is exactly given by $w_{\underline{R}}^{\mathrm{ICE}} = \exp[D_{\underline{\alpha}}(\vec{P}_X)]$, where $\underline{R} = (R_1, \ldots, R_d)$ is a risk-aversion vector with $R_k = 1 + \alpha_k/\alpha_0$ being the risk-aversion parameter associated with lottery k. Furthermore, we introduce a new conditional multivariate Rényi divergence that characterises a generalised scenario where the rational agent is allowed to have access to side information. We prove that this new quantity satisfies a data processing inequality which can be interpreted as the increment in the economic-theoretic value provided by side information; crucially we show that such a data processing inequality is a consequence of the agent's economic-theoretically consistent risk-averse attitude towards every lottery and vice versa. Moreover, we demonstrate the applicability of these results to the resource theory of informative measurements within the operational framework

Maxwell's Demon walks into Wall Street: Stochastic Thermodynamics meets Expected Utility Theory

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(Dated: December 14, 2023)

The interplay between thermodynamics and information theory has a long history, but its quantitative manifestations are still being explored. We import tools from expected utility theory from economics into stochastic thermodynamics. We prove that, in a process obeying Crooks' fluctuation relations, every α Rényi divergence between the forward process and its reverse has the operational meaning of the "certainty equivalent" of dissipated work (or, more generally, of entropy production) for a player with risk aversion $r = \alpha - 1$. The two known cases $\alpha = 1$ and $\alpha = \infty$ are recovered and receive the new interpretation of being associated to a risk-neutral and an extreme risk-averse player respectively. Among the new results, the condition for $\alpha = 0$ describes the behavior of a risk-seeking player willing to bet on the transient violations of the second law. Our approach further leads to a generalized Jarzynski equality, and generalizes to a broader class of statistical divergences.

Cedric Bleuler, Amos Lapidoth. Conditional Renyi divergences and horse betting, Entropy, 2020.

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