A new approximate Eastin-Knill theorem

Rhea Alexander

npj Quantum Inf ***12**, 15 (2026)*

UNIVERSIDAD DE GRANADA

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Part 1: What and why?

Part 2: Main results

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Part 1: What and why?

Part 2: Main results

Part 3: Methods: QEC + transversal gates = asymmetry distillation

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Part 1: What and why?

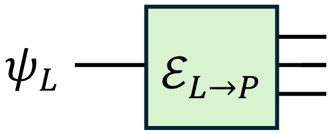
The goal…

fault-tolerant quantum computer

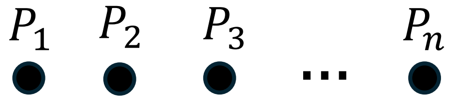
Image from: https://www.nag.co.za/2012/04/24/tech-quantum-computing-takes-a-leap-this-week/

Ingredients for fault-tolerance

1. **“Error resistant” quantum memory:** *QEC code*



encode



physical space

Campbell et al. *Nature* **549**, 172–179 (2017)

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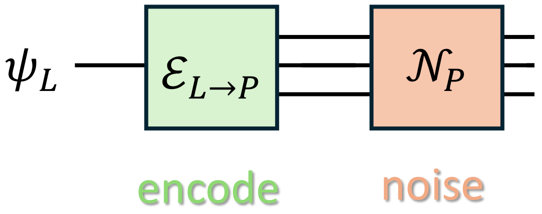

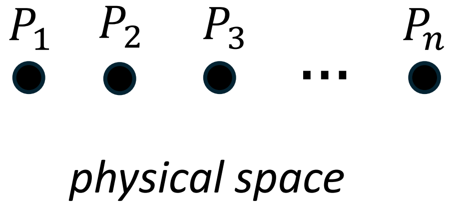

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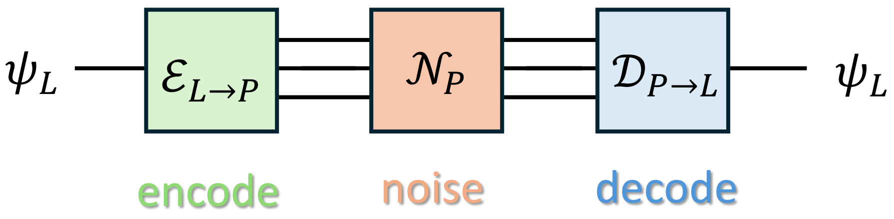
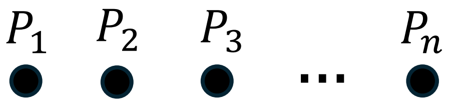


…for all ψ_L



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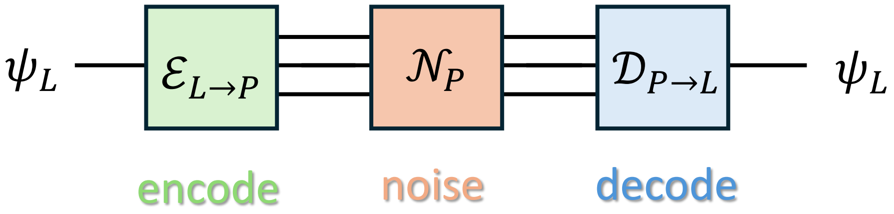

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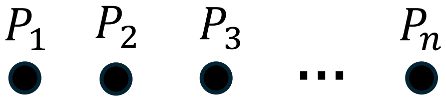

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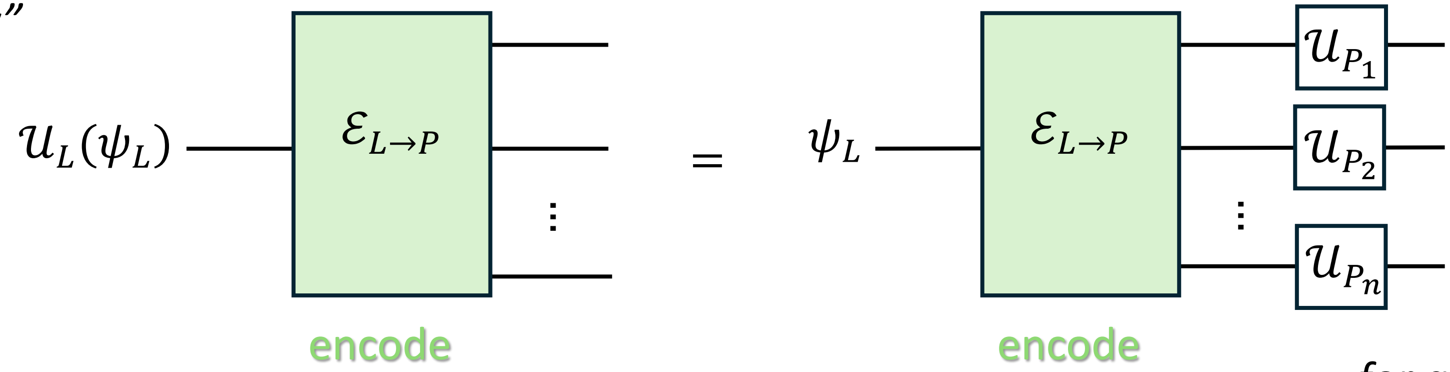
2. **"Error resistant" quantum processor:** *fault-tolerant logical gates*

Simplest: *"transversality"*



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Stumbling block

Eastin-Knill theorem: There is no (finite-dimensional) QEC code with a universal transversal gate set which *exactly* corrects for subsystem erasure[1]

[1] Eastin & Knill (2009); [2] Bény & Oreshkov (2010)

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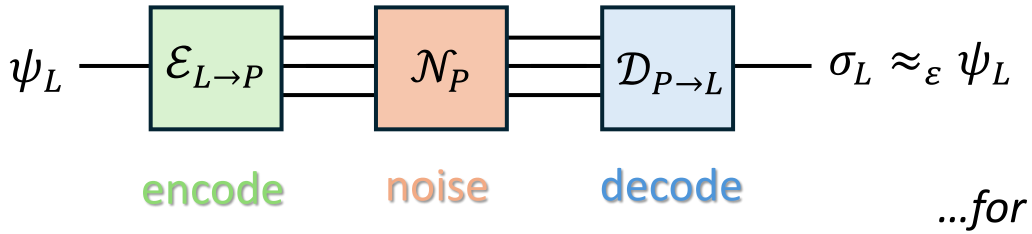
- **Approximate QEC:** relax assumption of *exact* error correction[2]

"ε-correctability"



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Approximate Eastin-Knill

- "Robust" variants of Eastin-Knill provide lower bounds on asymptotic QEC inaccuracy of codes with UTG (<u>necessary conditions</u>)[1]

$$\varepsilon = \Omega(1/n^2)$$

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- **Achievability?**
  - Explicit constructions of approximate codes with UTG [2] (sufficient conditions):
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Q: *necessary <u>and</u> sufficient conditions for single-shot regime?*

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Part 2: Main results

Main result

Theorem. *Any encoder $\mathcal{E}_{L\to P}$ with a universal set of transversal gates is ε-correctable w.r.t erasure of m subsystems $\mathcal{N}_{P\to P'} := \mathrm{tr}_{P_1 \dots P_m}$* iff

$$H_{min}(L|P')_{J(\mathcal{N}\circ\mathcal{E})} \leq -\log d_L \,(1 - c_L \varepsilon)$$

$$c := \frac{d+1}{d}$$

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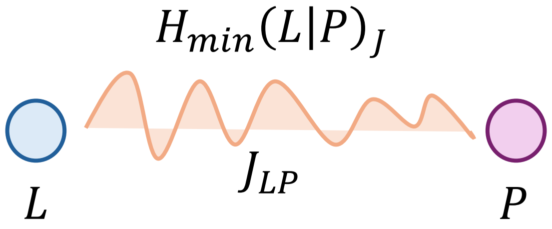

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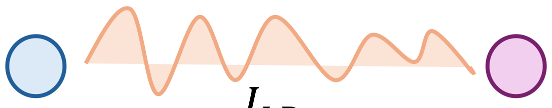

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- *N.B.* Global min $H_{min}(L|P)$ of $-\log d_L$ achieved for **maximally entangled states** over *LP*

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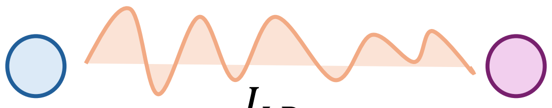

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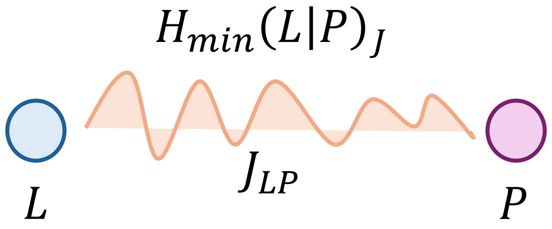

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$$\Rightarrow \varepsilon \geq \frac{1}{d_L(d_L+1)}\frac{1}{n^2} \quad (n \gg 1)$$

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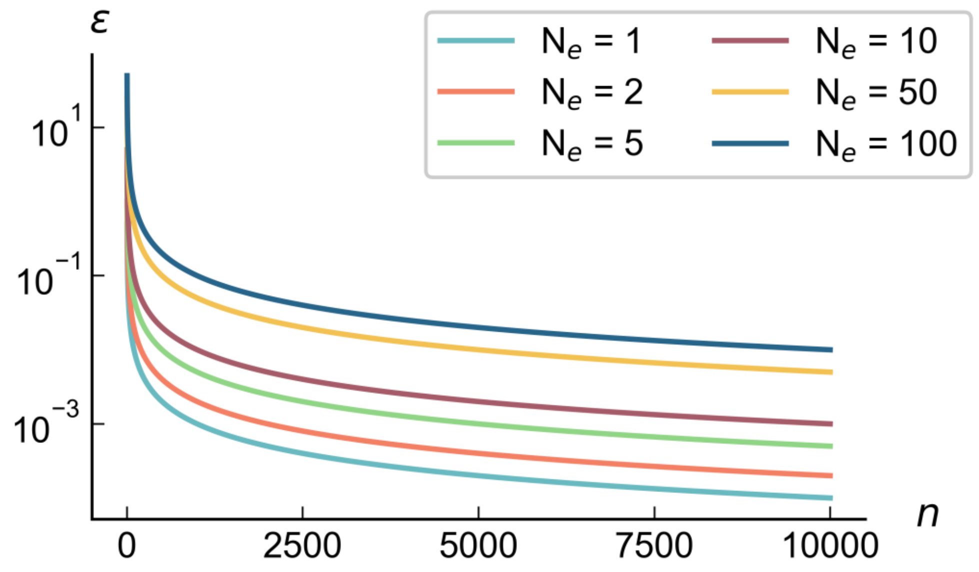
Example: W-state code

$$|\psi_L\rangle \to \frac{1}{\sqrt{n}}\left(|\psi,\perp,\perp,\dots,\perp\rangle + |\perp,\psi,\perp,\dots,\perp\rangle + \cdots |\perp,\perp,\perp,\dots,\psi\rangle\right), \qquad \langle\psi|\perp\rangle = 0$$

$$U_L \to U_L^{\otimes n}$$



The W-state code can ε-correct erasure of N_e subsystems iff

$$\varepsilon \geq \frac{N_e}{n}\left(1 - \frac{1}{d_L}\right)$$

e.g. n = 100 qutrits can encode 1 logical qubit and correct for single subsystem erasure up to ε = 0.005

Optimization-free metric

Define "near-optimal infidelity" [1]:

$$\delta(\mathcal{N} \circ \mathcal{E}) \coloneqq c_L^{-1}(1 - d_L^{-2}\|\mathrm{tr}_L\sqrt{M}\|_2^2)$$

QEC matrix M:

$$M_{[i,n][j,m]} \coloneqq \langle i_L | N_n^\dagger N_m | j_L \rangle$$

- $\{N_m\}$ *Kraus ops of* $\mathcal{N}$
- $\{|j_L\rangle\}$ *code words of* $\mathcal{E}$

(= data needed for Knill-Laflamme)

Corollary. *Any encoder $\mathcal{E}_{L\to P}$ with a universal transversal set of gates can ε-correct for erasure of m subsystems…*

if: $\varepsilon \geq \delta\left(\mathrm{tr}_{P_1\ldots P_m} \circ \mathcal{E}\right)$

only if: $\varepsilon \geq \frac{1}{2}\delta(\mathrm{tr}_{P_1\ldots P_m} \circ \mathcal{E})$

cost: $O\left(\left(d_L \prod_{i=1}^{m} d_{P_i}\right)^3\right)$ *vs.* $O\left((d_L d_P)^{5.246}\right)$

[1] Zeng et al. *Phys. Rev. Lett.* **132**, 250602 (2024).

Part 3: Methods

Methods

1) *For G-covariant noise $\mathcal{N}$ and encoder $\mathcal{E}$, optimal decoding achieved by G-covariant channel [1]*

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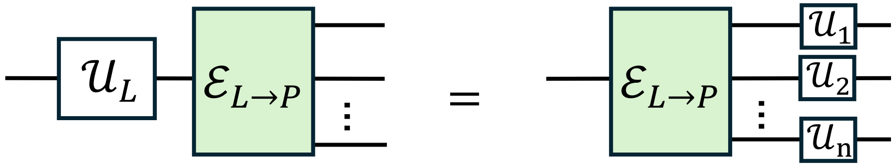

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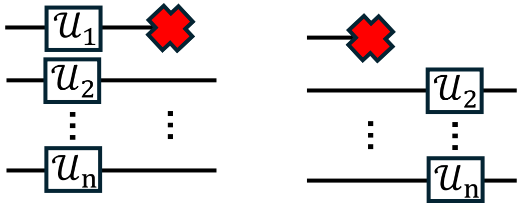

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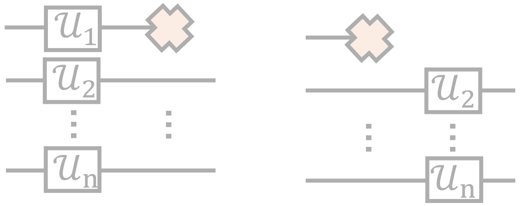

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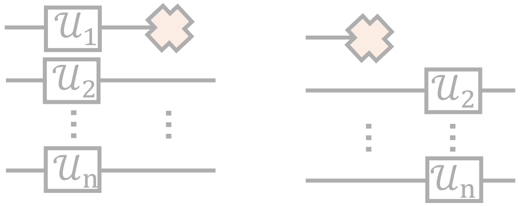

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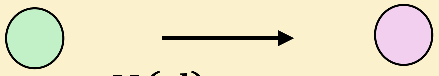
⇒ QEC as asymmetry distillation task

$\rho \coloneqq \mathcal{N} \circ \mathcal{E}(\psi_L)$ $\approx_\varepsilon \psi_L$



$U(d) - cov$

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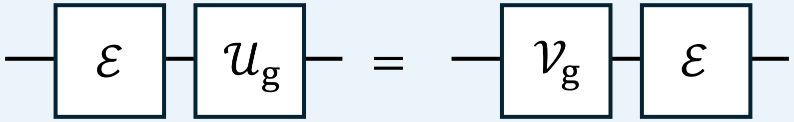
Resource theory of asymmetry

☆ Free operations = ***G*-covariant** channels



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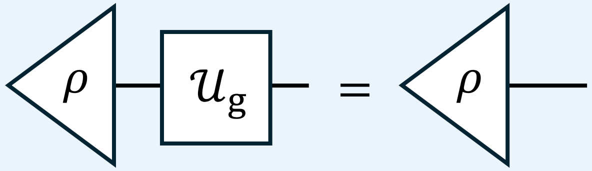

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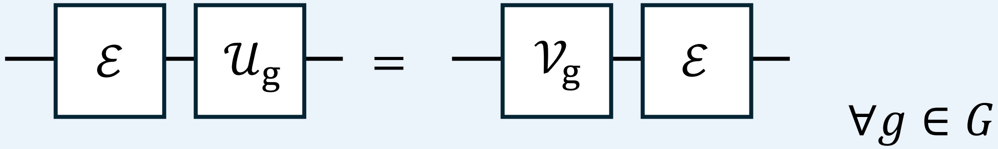

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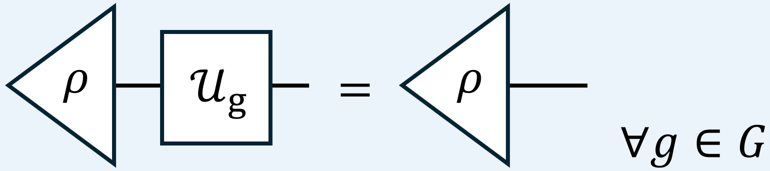
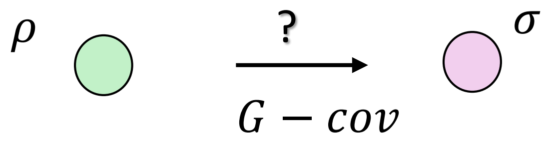


Q: *What can we say about future dynamics of a system in initial configuration ρ by appealing to symmetries?*



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Resource theory of asymmetry

$\phi \xrightarrow{\mathcal{V}_{\mathrm{G}}} \psi \quad \Leftrightarrow \quad \mathrm{tr}[\phi f(L)] = \mathrm{tr}[\psi f(L)], \forall$ *continuous f, generators L*

[Noether 1918]

[1] Noether *Nachr. König. Gesellsch. Wiss. Göttingen* **2**, 235 (1918); [2] Gour *et al.* *Nat Commun* **9**, 5352 (2018); [3] Marvian *Nat commun* **11**, 25 (2020)

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[Noether 1918]

ρ → $\mathcal{E}_{\mathrm{G}}$ → σ $\Leftrightarrow$ $H_\eta(\rho) \leq H_\eta(\sigma), \forall$ *states* η [Gour *et al.* 2018]

$H_\eta(\rho) \coloneqq H_{min}(R|A)_{\Pi_G(\eta^T \otimes \rho)}$

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$\rho \xrightarrow{\mathcal{E}_{\mathrm{G}}} \psi \quad \Leftrightarrow \quad H_\psi(\rho) \leq H_\psi(\psi)$ [Marvian 2020]

$\forall \lambda \in \Lambda$: $\rho_\lambda \xrightarrow{\mathcal{E}_{\mathrm{G}}} \approx_\varepsilon \psi_\lambda \quad \Leftrightarrow \quad \forall \{p_\lambda\}$: $H_{min}(R|A)_{\Pi_G(\sum_{\lambda \in \Lambda} p_\lambda \psi_\lambda^T \otimes \rho_\lambda)} \geq -\log(1-\varepsilon)$ **[this work]**

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ϕ ○ $\xrightarrow{\mathcal{V}_G}$ ○ ψ $\Leftrightarrow$ $\mathrm{tr}[\phi f(L)] = \mathrm{tr}[\psi f(L)], \forall$ *continuous f, generators L*

[Noether 1918]

ρ ○ $\xrightarrow{\mathcal{E}_G}$ ○ σ $\Leftrightarrow$ $H_\eta(\rho) \leq H_\eta(\sigma), \forall$ *states* η [Gour *et al.* 2018]

$$H_\eta(\rho) \coloneqq H_{min}(R|A)_{\Pi_G(\eta^T \otimes \rho)}$$

ρ ○ $\xrightarrow{\mathcal{E}_G}$ ○ ψ $\Leftrightarrow$ $H_\psi(\rho) \leq H_\psi(\psi)$ [Marvian 2020]

$\forall \lambda \in \Lambda$:

ρ_λ ○ $\xrightarrow{\mathcal{E}_G}$ ○ $\approx_\varepsilon \psi_\lambda$ $\Leftrightarrow$ $\forall \{p_\lambda\}$: $H_{min}(R|A)_{\Pi_G(\sum_{\lambda \in \Lambda} p_\lambda \psi_\lambda^T \otimes \rho_\lambda)} \geq -\log(1-\varepsilon)$ **[this work]**

symmetry + additional linearity constraints

[1] Noether *Nachr. König. Gesellsch. Wiss. Göttingen* **2**, 235 (1918); [2] Gour *et al.* *Nat Commun* **9**, 5352 (2018); [3] Marvian *Nat commun* **11**, 25 (2020)

Take home message – future work

☆ (Single-shot) entropic constraint on unitarily-covariant codes ☆

- *Other symmetries? E.g. Clifford group [**no-go** Chakraborty-Gottesman 2026] / permutation invariant codes…*
- *Noise models beyond erasure [Gupta et al. 2024]?*
- *Explicit construction of efficient (?) optimal decoders for covariant codes*
- *Asymptotic analysis?*
- *Minimal supersets of covariant encoders for circumventing E-K a la [Liu & Zhou 2023]*

☆ Thanks!! ☆

Quantum thermodynamics and computing Granada (QTCG)

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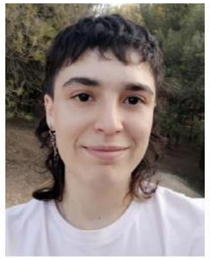

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